MODEL FIELD THEORIES

Thesis

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HANS KONRAD GRIMMER
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DR. ROBERT SCHLAPP

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All the material in this thesis is original except where explicit reference has been made.

CONTENTS

I INTRODUCTION AND DESCRIPTION OF CONTENTS

This thesis will discuss a number of model field theories. I think that to many high energy physicists it will look more like a mathematical than a physical investigation since physical results just guided me in choosing my assumptions and questions. The answers were then determined by mathematics, not physical intuition. And even in choosing assumptions and selecting questions limits were imposed by my finite knowledge of mathematical techniques. I am convinced however that both the mathematically careful and the phenomenological approaches to physics are necessary for progress in our understanding of physical phenomena. I hope that, some time, we shall also in high energy physics arrive at a mathematically consistent theory that accounts for the basic observed phenomena and which allows us to compute the more complicated ones if we are patient enough.

I had two main motives for studying my model field theories. First I wanted to understand better some difficulties that arise when one tries to find relativistic field theories describing the behaviour of the experimentally known particles. For this Klauder's so called "rotationally-symmetric model field theories" (RS-models) provide a good starting point. They are described in references 1, 2, and 3. Secondly I wanted to find related models closer to physical reality in order to see what features of RS-models should survive in future realistic field theories for high-energy phenomena and to see whether RS-models might be a starting point for constructing such theories.

I will now briefly sketch the contents of the various chapters. In the next the weak correspondence principle (WCP) will be discussed. It states the relation that must exist between a Euclidean, i.e. translation and rotation invariant quantum field theory and its classical counterpart. The RS-models will provide an excellent example for many of its unfamiliar features. The WCP will serve me as a guideline for defining my own models, which do not show full Euclidean invariance. This is reasonable since all the results for RS-models that we shall present will remain valid if we restrict the field to a finite region in space and thus violate Euclidean invariance.

In the third chapter, we shall define Klauder's rotationally-symmetric model field theories and state some of the results that he obtained for them. They are quite strong and will serve as an ideal for the kind of results I shall try to deduce in later chapters for more complicated models.

The fourth chapter deals with certain aspects of the reducibility of Hilbert spaces in which a linear group is represented. It is of a purely mathematical nature and will provide us with tools that will be applied in the following chapter to a further discussion of RS-models. The investigation is however kept general enough to serve us later also in dealing with the Hamiltonians of other model field theories.

The fifth chapter contains new investigations about RS-models. They are an outgrowth of testing the methods I developed for dealing with more complicated models in the simple RS-case. The main achievements are summarized at the end of the chapter.

Two kinds of field theories, called C_1 and C_2, that are related to but more complicated than RS-models are considered in the following two chapters. One of their advantages from a physical point of view is that they can describe interactions closer to local ones than is possible in the RS-case. The Hamiltonians for C_1-models still have a point spectrum only but this shortcoming also is overcome in the C_2-models. For the C_1-theories we shall just quote the results since their derivation is similar and often somewhat easier than the proof of the corresponding properties in the C_2-case. The physical interpretation of C_2-models is discussed and a special model considered, which tries to imitate the relativistic free field as closely as is possible for C_2-models.

II THE WEAK CORRESPONDENCE PRINCIPLE

This chapter is mainly an account of work done by J.R. Klauder and published in reference 4. The exposition in this chapter will be heuristic, which will allow me to formulate the important points more elegantly and in a language more familiar to physicists.

1 The Usual Correspondence Principle

First I want to remind you of the canonical procedure to pass from classical to quantum mechanics for the case of one degree of freedom. One starts with the Hamiltonian formulation of the classical problem by giving a function H(p,q) where the variables satisfy

$$\dot{p} = -\frac{\partial H(p,q)}{\partial q} \;,\; \dot{q} = \frac{\partial H(p,q)}{\partial p} \quad .$$

More generally the time development of any function A(p,q) that does not explicitly depend on time is given by

$$\dot{A}(p,q) = \left(\frac{\partial A}{\partial q}\frac{\partial H}{\partial p}-\frac{\partial H}{\partial q}\frac{\partial A}{\partial p}\right) = \{A,H\} \quad ,$$

which is called the Poisson bracket of A and the Hamiltonfunction H. The quantization prescription is then to replace p and q by operators $\underline{p}$ and $\underline{q}$ such that the Poisson bracket {A,B} of two functions of p and q goes over into $-i[\underline{A},\underline{B}]$, where $[\underline{A},\underline{B}]$ is the commutator of the two operators $\underline{A}$ and $\underline{B}$ corresponding to A and B.*

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* This program cannot be carried out fully. R.Arens and D.Babbitt have shown in reference 5 that for any classical Hamiltonian of the form $H=\frac{1}{2}p^2+f(q)$ {A,B} cannot go over into $-i[\underline{A},\underline{B}]$ for all A and B, not even if we restrict our attention to polynomials in p and q. I am indepted to Prof. R.F.Streater for pointing this out to me.

Since $\{q,p\} = 1$, the operators $\underline{q}$ and $\underline{p}$ e.g. will have to satisfy

$$[\underline{q},\underline{p}] = i \quad .$$

The situation is similar for a field. The classical Hamiltonian is then a functional $H(\pi(\underline{x}),\phi(\underline{x}))$ where the field $\phi(\underline{x})$ and the momentum $\pi(\underline{x})$ are again canonically conjugate. To quantize one replaces $\pi(\underline{x})$ and $\phi(\underline{x})$ by operators satisfying the canonical commutation relations (CCR)

$$[\underline{\phi}(\underline{x}),\underline{\pi}(\underline{y})] = i\delta(\underline{x}-\underline{y}) \quad , \quad [\underline{\phi}(\underline{x}),\underline{\phi}(\underline{y})] = [\underline{\pi}(\underline{x}),\underline{\pi}(\underline{y})] = 0 \qquad 1$$

and imposes normal ordering. This prescription is called the correspondence principle (CP).

It is well known that the above procedure often leads into difficulties. The interaction Hamiltonian is in general just not a linear operator defined on a dense domain of the Hilbert space. Therefore, efforts have been made to define operator products at the same space-time point so as to attain an operator. But this can be shown to be impossible in general.

An example where the CP works for the most common operators is the relativistic free field. The classical Hamiltonian is in this case

$$H(\pi,\phi) = \frac{1}{2}\int\{\pi^2(\underline{x})+(\nabla\phi)^2(\underline{x})+m^2\phi^2(\underline{x})\}d\underline{x} \quad . \qquad 2$$

The corresponding quantum Hamiltonian will be

$$\mathcal{H} = \frac{1}{2}\int\{:\underline{\pi}^2(\underline{x}):+:(\nabla\underline{\phi})^2(\underline{x}):+m^2:\underline{\phi}^2(\underline{x}):\}d\underline{x} \quad . \qquad 3$$

Let us consider its matrix elements between the states

$$\Phi[\pi(\underline{x}),\phi(\underline{x})] = e^{i\int\{\underline{\phi}(\underline{x})\pi(\underline{x})-\underline{\pi}(\underline{x})\phi(\underline{x})\}d\underline{x}}\Phi_0 \quad , \qquad 4$$

where Φ_0 is the normalized ground state of this theory. One finds that

$$\left(\Phi[\pi,\phi],\ \mathcal{H}\Phi[\pi,\phi]\right) = H(\pi,\phi) \quad ; \qquad 5$$

i.e. the diagonal matrix elements of the quantum Hamiltonian yield the classical Hamiltonian.

2 The Weak Correspondence Principle

Eq. 5 gives the kind of relation between a classical and a corresponding quantum theory that is required by the weak correspondence principle (WCP). Klauder studied it in reference 4 e.g. for one neutral, scalar, Euclidean invariant, self interacting field and showed that it holds not only for the Hamiltonian but also for the six generators of Euclidean transformations.

To give the reader a better feeling for the content and importance of the WCP, let us look at its simplest manifestations. We shall from now on write the classical field and momentum as $g(\underline{x})$ and $f(\underline{x})$ so that instead of eq. 4 we have now

$$\Phi[f,g] = e^{i\int\{\underline{\phi}(\underline{x})f(\underline{x})-\underline{\pi}(\underline{x})g(\underline{x})\}d\underline{x}}\Phi_0 \quad . \qquad 6$$

As I mentioned, we assume the field to be neutral, i.e. $\underline{\phi}(\underline{x})$ and $\underline{\pi}(\underline{x})$ are hermitian, and scalar

$$e^{i\underline{a}\cdot\underline{\mathcal{P}}}e^{i\underline{\omega}\cdot\underline{\mathcal{J}}}e^{i\int\{\phi(\underline{x})f(\underline{x})-\pi(\underline{x})g(\underline{x})\}d\underline{x}}e^{-i\underline{\omega}\cdot\underline{\mathcal{J}}}e^{-i\underline{a}\cdot\underline{\mathcal{P}}}$$

$$= e^{i\int\{\phi(\underline{x})f(R^{-1}(\underline{x}-\underline{a}))-\pi(\underline{x})g(R^{-1}(\underline{x}-\underline{a}))\}d\underline{x}} \quad ,$$

where the self-adjoint generators of translations $\underline{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ and rotations $\underline{\mathcal{J}} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ satisfy the commutation relations

$$[\mathcal{P}_k, \mathcal{P}_l] = 0 \quad , \quad [\mathcal{J}_k, \mathcal{J}_l] = i\varepsilon_{klm}\mathcal{J}_m \quad , \quad [\mathcal{P}_k, \mathcal{J}_l] = i\varepsilon_{klm}\mathcal{P}_m$$

and commute with the self-adjoint Hamiltonian $\mathcal{H}$:

$$[\mathcal{P}_k, \mathcal{H}] = [\mathcal{J}_k, \mathcal{H}] = 0 \quad .$$

Further we assume that the Hilbert space $\mathscr{H}$ in which these operators act contains a vector Φ_0 with the properties

$$\mathcal{P}_k\Phi_0 = \mathcal{J}_k\Phi_0 = \mathcal{H}\Phi_0 = 0 \quad .$$

Klauder shows that it follows from these assumptions that for well behaved f and g

$$(\Phi[f,g], \mathcal{P}_k\Phi[f,g]) = \int f(\underline{x})\nabla_k g(\underline{x})d\underline{x}$$

and

$$(\Phi[f,g], \mathcal{J}_k\Phi[f,g]) = \int \varepsilon_{klm} f(\underline{x}) x_l \nabla_m g(\underline{x})d\underline{x} \quad ,$$

which are just the classical translation and rotation generators $P_k(f,g)$ and $J_k(f,g)$.

He further shows that operators satisfying the CP will always satisfy the WCP. However, if $\mathcal{P}_k$ and $\mathcal{J}_k$ are chosen according to the CP, they will often not satisfy the above assumptions, which are imparative on physical grounds. In these cases the WCP remains valid although the CP fails.

We speak of an irreducible representation of the CCR if every operator in $\mathcal{H}$ that commutes with all the $\underline{\phi}(\underline{x})$ and $\underline{\pi}(\underline{x})$ is a multiple of the identity. The Fock representation, which is the best known representation of the CCR, is irreducible. Irreducibility of the CCR representation is not implied by the WCP. This will be examplified by the RS-models.

In contrast to the usual correspondence principle, the WCP does not imply that the operators which satisfy it can be constructed from $\underline{\phi}$ and $\underline{\pi}$ only. In all the models that we shall consider, Φ_0 will be a nondegenerate eigenvector of H. A theorem due to Araki tells us then that for reducible CCR-representations H cannot be constructed from $\underline{\phi}$ and $\underline{\pi}$ only. This situation arises also for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ in the RS-models.

A special feature of our models will be that we shall maintain the CCR even in the interacting cases. We do not deny that many interacting models have been considered where the CCR are not satisfied. What we want to investigate is whether there are interesting models which do fulfil them.

III KLAUDER'S ROTATIONALLY-SYMMETRIC MODEL FIELD THEORIES

1 The Classical Models

We want to consider the models whose classical Hamiltonians are given by

$$H(f,g) = \frac{1}{2}[(f,f)+m_o^2(g,g)+V_o\{\zeta^2(g,g)\}] \quad , \qquad 7$$

where the two arguments of H, the field g and the momentum f are taken to be real valued elements of $L^2(\mathbb{R}_3)$, $f,g\epsilon L_r^2(\mathbb{R}_3)$. (f,f) and (g,g) on the r.h.s. of eq. 7 denote scalar products in $L^2(\mathbb{R}_3)$, linear in the second variable. For simplicity let us restrict V_o to polynomials

$$V_o\{\zeta^2(g,g)\} = \sum_{n=2}^{n_{max}} v_n\{\zeta^2(g,g)\}^n \quad , \qquad 8$$

where the summation starts with n=2 because we have collected all the linear terms in (g,g) in $m_o^2(g,g)$. Every $H(f,g)$ that can be written in the form of the r.h.s. of eq. 7 could be written in just one way were it not for the superfluous parameter ζ^2. We inserted ζ^2 because we shall find that to every classical model satisfying 7 and 8 and having $m_o>0$ there exists a one-parameter family of quantum theories corresponding to it in the sense of the WCP whenever $V_o\{\zeta^2(g,g)\}$ satisfies a certain positivity condition, that we shall spell out later.

Note that the Hamiltonian, eq. 7, is not relativistic: It lacks the gradient term and for non-vanishing V_o it is badly non-local, i.e. the relation between the values of the field at different points in space matters even if they are miles apart. $H(f,g)$ is invariant under a huge group of transformations in $L^2(\mathbb{R}_3)$ that map an element in $L^2_r(\mathbb{R}_3)$ into another in $L^2_r(\mathbb{R}_3)$, namely the ones which leave the scalar product invariant. We call this group $O(\infty,\mathbb{R}_3)$. O and ∞ refer to the fact that it is a group of (real) orthogonal transformations in an ∞-dimensional space and $\mathbb{R}_3$ reminds us that the integration involved in taking the scalar product which is left invariant extends over $\mathbb{R}_3$. The Euclidean transformations

$$(T\{\underline{a},R\}f)(\underline{x}) = f(R^{-1}(\underline{x}-\underline{a}))$$

are a locally-compact subgroup of $O(\infty,\mathbb{R}_3)$. Note that $H(f,g)$ is also invariant under "time reversal", i.e. under the replacement $f\to -f$, $g\to g$, which does not belong to $O(\infty,\mathbb{R}_3)$ since it affects only f, not g.

2 Smearing

We want to consider the quantum theories corresponding to our classical model, i.e. we are looking for a representation of the CCR in a separable Hilbert space $\mathcal{H}$ with positive definite scalar product. This is the space in which the quantum Hamiltonian $\mathcal{H}$ will act, and it must not be confused with the Hilbert space $L^2(\mathbb{R}_3)$ of which our classical field and momentum are elements.

We cannot expect $\underline{\phi}(\underline{x})$ and $\underline{\pi}(\underline{x})$ to be "good" operators in $\mathcal{H}$, i.e. linear operators defined on a dense domain in $\mathcal{H}$. This is clear already from the commutation relation, eq. 1, which involves a δ-function. Instead of $\phi(\underline{x})$ and $\pi(\underline{x})$ we shall therefore consider the " smeared" quantities

$$\phi(h) = \int \phi(\underline{x})h(\underline{x})d\underline{x} \quad \text{and} \quad \pi(h) = \int \pi(\underline{x})h(\underline{x})d\underline{x} \quad ,$$

where $h \in L^2(\mathbb{R}_3)$.

The CCR read now

$$[\phi(f), \pi(g)] = i(f,g) \quad , \qquad 9$$

and eq. 6 simplifies to

$$\Phi[f,g] = e^{i\{\phi(f)-\pi(g)\}}\Phi_o \quad . \qquad 10$$

3 Quantum Theory Assumptions

We group our assumptions into three sets dealing with the representation of the CCR, the ground state, and the Hamiltonian respectively.

(i) Representation of the CCR *

$\forall f \in L^2_r(\mathbb{R}_3)$ $\exists$ two self-adjoint operators $\phi(f)$ and $\pi(f)$ acting on a separable Hilbert space $\mathcal{H}$ with positive definite scalar product and satisfying

$$\phi(cf) = c\phi(f) \quad , \quad \pi(cf) = c\pi(f) \qquad \forall \text{ real } c \qquad 11$$

and such that

$$U[f,g] = e^{i\{\phi(f)-\pi(g)\}} \qquad 12$$

fulfils

$$U[f',g']U[f,g] = e^{\frac{i}{2}\{(f',g)-(g',f)\}}U[f'+f,g'+g] \quad . \qquad 13$$

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* We use the abbreviations $\forall$= for all and $\exists$= there exist(s), which are quite common in the mathematical literature.

The CCR, eq. 9, follow from eq. 13, which is called the Weyl form of the CCR. We prefer it because the $U[f,g]$ are unitary so that no domain questions arise.

(ii) Unique, cyclic, $O(\infty,\mathbb{R}_3)$-invariant state Φ_0

$\exists$ a vector $\Phi_0 \in \mathcal{H}$ such that finite linear combinations of vectors of the form

$$\Phi[f,g] = U[f,g]\Phi_0$$

are dense in $\mathcal{H}$.

$\forall T \in O(\infty,\mathbb{R}_3)$ $\exists$ a unitary transformation $U[T]$ with the property

$$U[T]\Phi[f,g] = \Phi[Tf,Tg] \quad , \qquad 14$$

and $\exists$ an antiunitary transformation $\mathcal{J}$ such that

$$\mathcal{J}\,\Phi[f,g] = \Phi[-f,g] \quad . \qquad 15$$

Up to a constant there is only one vector which is invariant under all the $U[T]$.

The first assumption, which can also be stated as "Every vector in $\mathcal{H}$ is a weak limit of finite linear combinations of the $\Phi[f,g]$'s, $\overline{\Phi[f,g]} = \mathcal{H}$", is called "cyclicity" of the representation of the CCR; Φ_0 is called a cyclic vector. Cyclicity is a weaker assumption than irreducibility: every irreducible representation is cyclic, and each vector of an irreducible representation is a cyclic vector. A reducible representation may be cyclic too, but not all its vectors can be cyclic.

The $\Phi[f,g]$ span $\mathcal{H}$, but in contrast to a basis they are not all linearly independent. We say that they form a total set or an overcomplete family of states (OFS).

The eqs. 15 and 14 determine $\mathcal{J}$ and U[T] thanks to the assumed antilinearity or linearity of these operators.

(iii) $O(\infty,\mathbb{R}_3)$-invariant Hamiltonian connecting ϕ and π as usual, with positive spectrum and unique ground state

$\exists$ a self-adjoint operator $\mathcal{H}\geq 0$ such that under suitable domain conditions

$$[U[T],\mathcal{H}] = 0 \qquad \forall\ T\in O(\infty,\mathbb{R}_3) \tag{16}$$

and

$$[\phi(f),\mathcal{H}] = i\pi(f) \quad . \tag{17}$$

Up to a constant the equation

$$\mathcal{H}\Phi = 0 \tag{18}$$

has a unique solution.

It is easy to show that the Φ in eq. 18 coincides with Φ_0.

I would like to point out the main features of our choice of assumptions. We assumed the existence of smeared field and momentum operators satisfying the CCR and we incorporated the huge and unrealistic $O(\infty,\mathbb{R}_3)$-invariance. If we required Poincaré invariance instead, i.e. invariance under the inhomogeneous Lorentz group, then our assumptions would all be fairly standard. Klauder insisted on $O(\infty,\mathbb{R}_3)$ invariance for computational convenience. In fact the main two theorems that he derived in reference 1 and that I am going to present below are surprisingly strong if we compare them with deductions that have been made, say, from the Wightman axioms.

4 Quantum Theory Results

We group the main results into two theorems. The first deals with the overlap $(\Phi_0,\Phi[f,g])$, which we will call the reproducing kernel*, and makes use of the first two bunches of assumptions only. The second will deal with the Hamiltonian.

Theorem 1**

> Every reproducing kernel satisfying (i) and (ii) is of the form
>
> $$(\Phi_0,\Phi[f,g]) = e^{-\frac{1}{4}\{\frac{\xi}{m}(f,f)+m(g,g)\}} \quad \text{with } m>0,\ \xi\geq 1 \quad . \qquad 19$$
>
> The pairs (m,ξ) satisfying the above restrictions are in one-to-one correspondence with the equivalence classes of CCR representations that satisfy (i) and (ii). The representations are irreducible if $\xi=1$, reducible if $\xi>1$.

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* The term reproducing kernel should really be given to the more general quantities $(\Phi[f^1,g^1],\Phi[f,g])$. However these are completely determined by $(\Phi_0,\Phi[f,g])$. Using eq. 13a it follows namely that

$$(\Phi[f^1,g^1],\Phi[f,g]) = e^{-\frac{i}{2}\{(f^1,g)-(g^1,f)\}}(\Phi_0,\Phi[f-f^1,g-g^1]) \quad .$$

** The CCR-representations appearing in theorem 1 are just those considered by Verboven in reference 6 for the Einstein model of a crystal. Our parameter m is there interpreted as the frequency ω of the oscillations of the atoms in the crystal and $\xi = (\mathrm{cth}\frac{\omega}{2kT})^2$. Verboven does not consider whether other representations would be possible too.

This theorem tells us about the existence of theories satisfying the first two sets of assumptions and gives a complete survey of them. Knowledge of the reproducing kernel characterizes them up to unitary equivalence.

Theorem 2

For each of the representations of theorem 1 $\exists$ a Hamiltonian satisfying (iii) and with matrix elements

$$(\Phi[f^1,g^1],H\Phi[f,g]) =$$
$$\frac{1}{2}[(f^1-img^1,f-img)+V\{\zeta^2(g^1,g)\}](\Phi[f^1,g^1],\Phi[f,g]) \quad , \tag{20}$$

where $\zeta^2 = \frac{\xi-1}{\xi}$ for every $V\{z\} = \sum_{n=1}^{n_{max}} v_n z^n$ such that $\sum_{n=1}^{n_{max}} v_n(\frac{2}{m})^n \frac{q!}{(q-n)!} > 0$ for $q=1,2,...$

The last restriction is the promised positivity condition. It is manifestly fulfilled if all the $v_n>0$. We write

$$V\{z\} = v_1 z+V_0\{z\} \quad .$$

5 Discussion

In order to see to which classical model a given quantum Hamiltonian belongs, we look at its diagonal matrix elements:

$$(\Phi[f,g],H\Phi[f,g]) = \frac{1}{2}[(f,f)+m^2(g,g)+V\{\zeta^2(g,g)\}]$$
$$= \frac{1}{2}[(f,f)+m_0{}^2(g,g)+V_0\{\zeta^2(g,g)\}] \quad . \tag{21}$$

We have put $m_0{}^2=m^2+v_1\zeta^2$. Notice that $m_0{}^2\geq m^2>0$. Eq. 21 is just of the form of the classical Hamiltonian H(f,g) as the WCP requires.

Note that the case $\xi=1$ encompasses only "free" models, where $V_0\{\ \}\equiv 0$, i.e. if we want to have interaction we must employ a reducible representation. For all $\xi>1$ the same interactions are possible.

I would like to say a few words about the particle interpretation. Klauder et al. find in reference 2

$$(\Phi_0,\phi(f^1)e^{-iHt}\phi(f)\Phi_0) = \frac{1}{2}(f^1,f)\left[\frac{\rho_-}{m_-}e^{-im_-t}+\frac{\rho_+}{m_+}e^{-im_+t}\right]$$

with

$$\rho_\pm = \frac{m_\pm}{m_\pm - m_\mp}\left(1-\xi\frac{m_\mp}{m}\right) \quad \text{and}$$

$$m_\pm = \frac{m}{2}\left(1+V_1\pm\sqrt{(1-V_1)^2+4V_1\xi^2}\right), \quad V_1 = \frac{v_1}{m^2} .$$

Therefore the two-point function can heuristically be written as

$$(\Phi_0,\phi(\underline{x},t)\phi(\underline{y})\Phi_0) = \frac{1}{2}\delta(\underline{x}-\underline{y})\left[\frac{\rho_-}{m_-}e^{-im_-t}+\frac{\rho_+}{m_+}e^{-im_+t}\right] ,$$

which shows that our models possess in the reducible case two asymptotic one-particle states and just one in the irreducible case.

The two theorems have been derived without the introduction of boxes, momentum space cut-offs, or perturbation expansions. At no stage of the calculations did divergencies arise.

It is nevertheless interesting to investigate whether these models could also be solved by perturbation theoretic methods. This has been undertaken by J.G.Taylor in reference 2 for the special case

$$H(f,g) = \frac{1}{2}[(f,f)+m_o^2(g,g)+\lambda(g,g)^2] \quad .$$

He found that mass, coupling constant, and wave function renormalization cannot remove all divergencies. The additional asymptotic one-particle state in reducible representations gets lost. The only case where the perturbation procedure works is $\lambda=0$, i.e. the free case. These results are not surprising: A cut-off had to be introduced and the Fock representation, which is irreducible, was used. The final removal of the cut-off does not make the representation reducible, and, therefore, meaningful results can be expected in the case $\lambda=0$ only.

IV ON THE REDUCIBILITY OF REPRESENTATIONS OF LINEAR GROUPS

1 Tensors

I found many of the results that I shall present in this chapter first in a more pedestrian way. But recently I became aware of their close resemblence with corresponding results that Hamermesh derives in reference 7 for GL(N) by studying irreducible tensors with respect to this group and by using Young diagrams. I shall in this chapter, therefore, use this more elegant and to many familiar language.

We introduce an orthonormal basis $a_1, a_2, \dots$ in $L^2(\mathbb{R}_3)$. Then we can associate with each $h \in L^2(\mathbb{R}_3)$ a vector $(h^1, h^2, \dots)$ in l^2, where $h^i = (a_i, h)$. $\sum_{i=1}^{\infty} h^{i*} h^i$ is finite and equals (h,h). Consider a group $\mathcal{T}$ of bounded linear transformations in $L^2(\mathbb{R}_3)$, e.g. $\mathcal{T} = O(\infty, \mathbb{R}_3)$. The transformations $T \in \mathcal{T}$ change h into $\tilde{h}$, which can be written in the language of l^2 as

$$\tilde{h}^i = \sum_{j=1}^{\infty} T^i_j h^j \quad , \quad i = 1, 2, \dots \quad . \tag{22}$$

Let $h_1, h_2, \dots, h_r \in L^2(\mathbb{R}_3)$ and consider the quantities $h_1^{i_1} \cdot h_2^{i_2} \cdot \dots \cdot h_r^{i_r}$.

$$\sum_{i_1, \dots, i_r = 1}^{\infty} (h_1^{i_1} \cdot \dots \cdot h_r^{i_r})^* (h_1^{i_1} \cdot \dots \cdot h_r^{i_r}) = \prod_{n=1}^{r} \left(\sum_{i_n = 1}^{\infty} h_n^{i_n *} h_n^{i_n} \right)$$

is finite since each of the r factors on the r.h.s. is.

When the h_n , n=1,...,r are transformed according to eq. 22, $h_1^{i_1}\ldots h_r^{i_r}$ will go into

$$\tilde{h}_1^{i_1}\cdot\ldots\cdot\tilde{h}_r^{i_r} = \sum_{j_1,\ldots,j_r=1}^{\infty} T_{j_1}^{i_1}\ldots T_{j_r}^{i_r}\, h_1^{j_1}\cdot\ldots\cdot h_r^{j_r} \quad . \tag{23}$$

More generally we will call a set of quantities $F^{i_1\ldots i_r}$ a tensor of rank r with respect to the group $\mathcal{T}$ if

$$\tilde{F}^{i_1\ldots i_r} = \sum_{j_1,\ldots,j_r=1}^{\infty} T_{j_1}^{i_1}\ldots T_{j_r}^{i_r}\, F^{j_1\ldots j_r} \quad . \tag{24}$$

We abbreviate this as

$$\tilde{F}^{(i)} = \sum_{(j)} T_{(j)}^{(i)}\, F^{(j)} \quad . \tag{24'}$$

2 Permutations

The permutations p of r elements $\{1,2,\ldots,r\}\to\{a_1,a_2,\ldots,a_r\}$, which we write as

$$p = \binom{1\;2\;\ldots r}{a_1a_2\ldots a_r} \quad ,$$

form a group S_r, called the symmetric group of degree r. Obviously

$$T_{j_1}^{i_1}\ldots T_{j_r}^{i_r} = T_{j_{a_1}}^{i_{a_1}}\ldots T_{j_{a_r}}^{i_{a_r}} \quad ,$$

which we abbreviate as

$$T_{(j)}^{(i)} = T_{p(j)}^{p(i)} \quad .$$

We associate with each $p\in S_r$ an operator $\underline{p}$ which replaces the 1st index of a tensor by the a_1th, the 2nd by the a_2th, etc.:

$$(\underline{p}F)^{i_1\ldots i_r} = F^{i_{a_1}\ldots i_{a_r}}$$

or in abbreviated notation

$$(\underline{p}F)^{(i)} = F^{p(i)} \quad .$$

I want to show that the $\underline{p}$ commute with all the transformations 24':

$$(\underline{p}\tilde{F})^{(i)} = (\tilde{F})^{p(i)} = \sum_{(j)} T^{p(i)}_{p(j)} F^{p(j)} = \sum_{(j)} T^{p(i)}_{p(j)} (\underline{p}F)^{(j)}$$
$$= \sum_{(j)} T^{(i)}_{(j)} (\underline{p}F)^{(j)} \qquad 25$$

This commutation property implies that all tensors of rank r with a particular symmetry under permutation of their indices will be transformed among themselves by the transformations 24'. The space of r^{th} rank tensors is, therefore, reducible into subspaces consisting of tensors of different symmetry.

We denote by $(a_1,a_2,\ldots,a_n)$ the operator which brings, for $n>1$, the index that was at the $a_1{}^{th}$ place from the left in $F^{i_1\ldots i_r}$ to the $a_2{}^{th}$ place, the one that was at the $a_2{}^{th}$ to the $a_3{}^{th}$,..., the one that was at the $a_n{}^{th}$ to the $a_1{}^{th}$. For $n=1$, (a_1) shall be the identity operator. We call $(a_1,a_2,\ldots,a_n)$ a cycle. $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ commute if none of the a_i's equals any of the b_j's. Every permutation operator $\underline{p}$ can be written as a product of cycles that involves each of the numbers $1,\ldots,r$ just once. We shall omit cycles containing a single number. If the number of cycles with an even number of entries is even, then we will say that the permutation has parity $\sigma=+1$, if it is odd we will say that the parity $\sigma=-1$.

3 The Young Theory

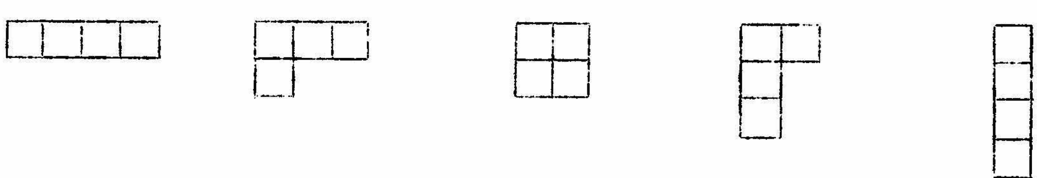
Young's procedure of splitting the space of all tensors into mutually orthogonal subspaces, each consisting of tensors with definite symmetry properties, is described in many textbooks (e.g. references 7 and 8). I will quote the result for a tensor of arbitrary rank r and illustrate it for special cases.

For each way of splitting r into a sum of nonnegative integers, that we order according to decreasing magnitude

$$\lambda_1+\lambda_2+\ldots+\lambda_r = r \quad , \quad \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_r\geq 0 \quad ,$$

we draw λ_1 squares in a line, λ_2 in the next below, then λ_3, etc., always so that the left ends coincide. These pictures are called the Young graphs. For r=4 we have the following ones:

4+0+0+0=4 3+1+0+0=4 2+2+0+0=4 2+1+1+0=4 1+1+1+1=4



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Fill one of the squares with the number 1, then another with 2, and so on until finally the number r is inserted into the last square. Do this in such a way that at each stage there are no unfilled squares to the left or above a filled one. In this manner we get the so called standard Young tableaux.

For r=4 they are

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline\end{array}\qquad \begin{array}{|c|c|}\hline 1&3\\ \hline 2&4\\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&4\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array}$$ 27

Each Young tableau stands for a symmetrization procedure, which I want to describe now.

Take the sum of all the operators corresponding to a permutation of the numbers in one of the rows. Let $\underline{P}$ denote the product of the sums for all the rows in the tableau. Similarly take the sum of all the operators corresponding to a permutation of the numbers in one of the columns, each multiplied with the parity of the permutation. Denote the product of the sums for all the columns in the tableau by $\underline{Q}$. $\underline{Y}=\underline{Q}\,\underline{P}$ is called the Young operator corresponding to the tableau.

Example 1: $\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline\end{array}$

$$\underline{P} = [e+(12)][e+(34)] \quad , \quad \underline{Q} = [e-(13)][e-(24)] \quad ,$$

(e is the identity),
i.e.

$$(\underline{P}F)^{i_1 i_2 i_3 i_4} = F^{i_1 i_2 i_3 i_4} + F^{i_2 i_1 i_3 i_4} + F^{i_1 i_2 i_4 i_3} + F^{i_2 i_1 i_4 i_3}$$

$$=$$

and $F_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}{}^{i_1i_2i_3i_4} = (\underline{Q}\underline{P}F)^{i_1i_2i_3i_4}$

$$=F^{i_1i_2i_3i_4}-F^{i_3i_2i_1i_4}-F^{i_1i_4i_3i_2}+F^{i_3i_4i_1i_2}$$

$$+F^{i_2i_1i_3i_4}-F^{i_3i_1i_2i_4}-F^{i_2i_4i_3i_1}+F^{i_3i_4i_2i_1}$$

$$+F^{i_1i_2i_4i_3}-F^{i_4i_2i_1i_3}-F^{i_1i_3i_4i_2}+F^{i_4i_3i_1i_2}$$

$$+F^{i_2i_1i_4i_3}-F^{i_4i_1i_2i_3}-F^{i_2i_3i_4i_1}+F^{i_4i_3i_2i_1}$$

Example 2: $\begin{smallmatrix}1&3\\2&\\4&\end{smallmatrix}$

$\underline{P}$ = e +(13) , $\underline{Q}$ = e-(12)-(24)-(41)+(124)+(142) , thus

$F_{\begin{smallmatrix}1&3\\2&\\4&\end{smallmatrix}}{}^{i_1i_2i_3i_4} = (\underline{Q}\underline{P}F)^{i_1i_2i_3i_4}$

$$=F^{i_1i_2i_3i_4}-F^{i_2i_1i_3i_4}-F^{i_1i_4i_3i_2}-F^{i_4i_2i_3i_1}$$

$$+F^{i_4i_1i_3i_2}+F^{i_2i_4i_3i_1}+F^{i_3i_2i_1i_4}-F^{i_2i_3i_1i_4}$$

$$-F^{i_3i_4i_1i_2}-F^{i_4i_2i_1i_3}+F^{i_4i_3i_1i_2}+F^{i_2i_4i_1i_3}$$

The symmetrization procedures corresponding to the standard tableaux for a given r lead all to mutually orthogonal subspaces; i.e. if we apply $\underline{Y}_\tau$ corresponding to the standard tableau τ to a tensor $F^{i_1\dots i_r}$ and then $\underline{Y}_{\tau^1}$ corresponding to the different standard tableau τ^1 to the result, we will get zero for every choice of $F^{i_1\dots i_r}$. However acting with $\underline{Y}_\tau$ again on $(\underline{Y}_\tau F)^{i_1\dots i_r}$ amounts just to a multiplication with a number c_τ that depends on the graph of τ

$$c_\tau = \frac{\text{(number of squares in the graph)!}}{\text{number of standard tableaux corresponding to the graph}} \cdot$$

If we normalize the Young operators by dividing them by c_τ,

$$Y_\tau = \frac{\underline{Y}_\tau}{c_\tau} ,$$

then the sum of the symmetrized F corresponding to the different standard tableaux will give $F^{i_1 \dots i_r}$ back,

$$\sum_\tau (Y_\tau F)^{i_1 \dots i_r} = F^{i_1 \dots i_r} .$$

An arbitrary tensor of rank r, $F^{i_1 i_2 \dots i_r}$, has r! elements with an index 1, an index 2,..., and an index r because there are r! permutations of r distinct elements. We expect that there are relations between these r! elements in the case of a symmetrized tensor and that the sum of the number of independent components of all the tensors corresponding to the different tableaux is r! In fact, the number of independent components of a tensor corresponding to a given standard tableau is equal to the number of different standard tableaux that belong to the graph in question. Let us check this for fourth rank tensors. Counting the tableaux in 27, the quoted result tells us that there are

$$1 \cdot 1 + 3 \cdot 3 + 2 \cdot 2 + 3 \cdot 3 + 1 \cdot 1 = 24$$

independent components.

Let us verify some of the above statements in the case of third rank tensors. The standard tableaux and the corresponding $\underline{Y}$'s and c_τ's are

τ	$\underline{Y}_\tau$	c_τ
$\begin{smallmatrix}1&2&3\end{smallmatrix}$	$\underline{Y}_{\begin{smallmatrix}1&2&3\end{smallmatrix}}=e+(12)+(23)+(31)+(123)+(132)$	$\rightarrow c_{\begin{smallmatrix}1&2&3\end{smallmatrix}}=6=\frac{3!}{1}$
$\begin{smallmatrix}1&2\\3\end{smallmatrix}$	$\underline{Y}_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} = [e-(13)]\;[e+(12)]$	$\rightarrow c_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}=3=\frac{3!}{2}$
$\begin{smallmatrix}1&3\\2\end{smallmatrix}$	$\underline{Y}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} = [e-(12)]\;[e+(13)]$	$\rightarrow c_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}=3=\frac{3!}{2}$
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\underline{Y}_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}=e-(12)-(23)-(31)+(123)+(132)$	$\rightarrow c_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}=6=\frac{3!}{1}$

$$(Y_{\begin{smallmatrix}1&2&3\end{smallmatrix}}F)^{i_1i_2i_3}=\frac{1}{6}\{\,F^{i_1i_2i_3}+F^{i_2i_1i_3}+F^{i_1i_3i_2}+F^{i_3i_2i_1}+F^{i_3i_1i_2}+F^{i_2i_3i_1}\}$$

$$(Y_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}F)^{i_1i_2i_3}=\frac{1}{6}\{2F^{i_1i_2i_3}+2F^{i_2i_1i_3}-2F^{i_3i_2i_1}-2F^{i_3i_1i_2}\}$$

$$(Y_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}F)^{i_1i_2i_3}=\frac{1}{6}\{2F^{i_1i_2i_3}-2F^{i_2i_1i_3}+2F^{i_3i_2i_1}-2F^{i_2i_3i_1}\}$$

$$(Y_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}F)^{i_1i_2i_3}=\frac{1}{6}\{\,F^{i_1i_2i_3}-F^{i_2i_1i_3}-F^{i_1i_3i_2}-F^{i_3i_2i_1}+F^{i_3i_1i_2}+F^{i_2i_3i_1}\}$$

$$\sum_\tau (Y_\tau F)^{i_1i_2i_3}=F^{i_1i_2i_3}$$

Abbreviate $(\underline{Y}_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}F)^{i_1i_2i_3}=G^{i_1i_2i_3}$. We expect two independent components among

$$G^{478},\ G^{487},\ G^{748},\ G^{784},\ G^{847},\ G^{874}\ .$$

In fact

$$G^{478} = F^{478}+F^{748}-F^{874}-F^{847}$$

$$G^{487} = F^{487}+F^{847}-F^{784}-F^{748}$$

$$G^{748} = F^{748}+F^{478}-F^{847}-F^{874} = G^{478}$$

$$G^{784} = F^{784}+F^{874}-F^{487}-F^{478} =-G^{478}-G^{487}$$

$$G^{847} = F^{847}+F^{487}-F^{748}-F^{784} = G^{487}$$

$$G^{874} = F^{874}+F^{748}-F^{478}-F^{487} =-G^{478}-G^{487}$$

Tensors that have been symmetrized according to procedures corresponding to different Young graphs span spaces containing inequivalent representations.

If we do not consider arbitrary tensors but restrict ourselves to those that have certain symmetry properties like being symmetric in the first two indices, then applying the Young operator to such a tensor may yield one that does not exhibit the initial symmetries and that, therefore, cannot be considered as the contribution to the original tensor from an invariant subspace. The sum of certain such "contributions" belonging all to the same graph, however, will exhibit the initial symmetries and will therefore in fact be a contribution from an invariant subspace. I shall return to this soon in more detail and more explicitely for cases of special interest.

Besides the tensors $F^{j_1 \cdots j_r}$, which transform according to eq. 24,

$$(U[T]F)^{i_1\ldots i_r} = \sum_{j_1,\ldots,j_r} T^{i_1}_{j_1}\ldots T^{i_r}_{j_r} F^{j_1\ldots j_r} ,$$

we can also introduce tensors $F_{i_1\ldots i_r}$ with the contravariant transformation properties

$$(U[T]F)_{i_1\ldots i_r} = \sum_{j_1,\ldots,j_r} T^{-1*j_1}_{\quad i_1}\ldots T^{-1*j_r}_{\quad i_r} F_{j_1\ldots j_r} .$$

As Young operators for the contravariant quantities we take

$$\underline{Y}^+ = (\underline{Q}\underline{P})^+ = \underline{P}^+\underline{Q}^+ .$$

By (F,G) we understand

$$\sum_{j_1\ldots j_r} F_{j_1\ldots j_r} G^{j_1\ldots j_r} .$$

We call (F,G) the scalar product of F and G. It has the properties

$(F,G)=(U[T]F,U[T]G)$ $\forall T\epsilon\mathcal{T}$ and $(\underline{Y}^+_{\tau^1}F,\underline{Y}_\tau G)=0$ for $\tau^1 \neq \tau$.

For T in an orthogonal group $\mathcal{T}$ there holds $T^{-1*j}_{\quad i} = T^i_j$, i.e. covariant and contravariant quantities transform in exactly the same manner, so that it will be convenient to write both with upper indices. The only difference between the two types is then that in one case we use $\underline{Y}$, in the other $\underline{Y}^+$ as Young operators. We can consider the first as kets, the second as bras.

4 Special Tensors

We shall now consider the tensors of special interest for us

$$F^{i_1 i_2\ldots i_r} = h^{i_1}_{a_1} h^{i_2}_{a_2}\ldots h^{i_r}_{a_r} = (h_{a_1}h_{a_2}\ldots h_{a_r})^{i_1 i_2\ldots i_r} .$$

Consider

$$(6\ 5\ 3\ 1)\ (h_{a_1} h_{a_2} h_{a_3} h_{a_4} h_{a_5} h_{a_6})^{796358}$$

$$=(h_{a_1} h_{a_2} h_{a_3} h_{a_4} h_{a_5} h_{a_6})^{695387} = (h_{a_6} h_{a_2} h_{a_1} h_{a_4} h_{a_3} h_{a_5})^{796358}\ .$$

The last form suggests that we can describe the action of $(6\ 5\ 3\ 1)$ as follows: Replace h_{a_6} by h_{a_5}, h_{a_5} by h_{a_3}, h_{a_3} by h_{a_1}, h_{a_1} by h_{a_6}. This last form has for our purposes many advantages over the middle one. We will no longer have to indicate the indices when writing down our sums and differences of tensors since they are equal for each term. In fact, we need no longer consider the h_i's as elements of l^2, we may consider them now again as elements of $L^2(\mathbb{R}_3)$ and speak of the variables of a tensor instead of speaking of its indices.

Let us apply $\underline{Y}_{\substack{1\,2\\3}}$ and $\underline{Y}_{\substack{1\,3\\2}}$ to various tensors.

F	$\underline{Y}_{\substack{\boxed{1}\boxed{2}\\\boxed{3}}}F$	$\underline{Y}_{\substack{\boxed{1}\boxed{3}\\\boxed{2}}}F$	
$h_1h_2h_3$	$h_1h_2h_3+h_2h_1h_3-h_3h_2h_1-h_2h_3h_1$	$h_1h_2h_3+h_3h_2h_1-h_2h_1h_3-h_3h_1h_2$	
$h_1h_1h_2$	$2h_1h_1h_2-h_2h_1h_1-h_1h_2h_1$	0	28
$h_1h_2h_2$	$h_1h_2h_2+h_2h_1h_2-2h_2h_2h_1$	$h_1h_2h_2+h_2h_2h_1-2h_2h_1h_2$	
$h_1h_1h_1$	0	0	

$h_1h_1h_2$ is symmetric in the first two variables and so is $\underline{Y}_{\substack{1\,2\\3}}h_1h_1h_2$. Similarly $h_1h_2h_2$ is symmetric in the last two variables, but neither $\underline{Y}_{\substack{1\,2\\3}}h_1h_2h_2$ nor $\underline{Y}_{\substack{1\,3\\2}}h_1h_2h_2$ are. The sum of the two, however, is symmetric in the last two variables. Another aspect of this situation is that the two are not independent,

$$(h_1h_2h_2+h_2h_1h_2-2h_2h_2h_1)^{i_1i_2i_3}$$

$$= (h_1h_2h_2-2h_2h_1h_2+h_2h_2h_1)^{i_1i_3i_2} \quad .$$

This outcome can be understood already by looking at Young tableaux: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ requires symmetrizing $h_{a_1}h_{a_2}h_{a_3}$ with respect to h_{a_1} and h_{a_2} and antisymmetrizing between h_{a_1} and h_{a_3}. If we replace 1,2, and 3 in $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and in $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ by a_1, a_2, and a_3 respectively, then we get in our example

$a_1a_2a_3$	$\begin{array}{\|c\|c\|}\hline a_1&a_2\\\hline a_3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline a_1&a_3\\\hline a_2\\\cline{1-1}\end{array}$
1 1 2	$\begin{smallmatrix}1&1\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\1&\end{smallmatrix}$
1 2 2	$\begin{smallmatrix}1&2\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\2&\end{smallmatrix}$

It can be seen from $\begin{smallmatrix}1&2\\1&\end{smallmatrix}$ that $\underline{Y}_{\substack{13\\2}}h_1h_1h_2=0$ since this tells us that we antisymmetrize between two identical quantities. That $\underline{Y}_{\substack{12\\3}}h_1h_2h_2$ and $\underline{Y}_{\substack{13\\2}}h_1h_1h_2$ are not independent follows from the fact that we have in both cases $\begin{smallmatrix}1&2\\2&\end{smallmatrix}$ indicating that we symmetrize in both cases between h_1 and one of the h_2 and antisymmetrize between h_1 and the other h_2.

More generally, we can proceed as follows to characterize the invariant subspaces that contribute to tensors which are constructed as products of vectors $h_{a_1}\cdot h_{a_2}\cdot\ldots\cdot h_{a_r}$ of which some are equal: we write down the different Young graphs and insert the numbers $a_1,\ldots,a_r$ into the squares in all possible standard ways. The meaning of "standard" has here been generalized to the requirement that the numbers do not decrease from left to right and do increase downwards.

Let us illustrate the method in a more complicated example. We want to split $h_1h_2h_2h_3$ into a sum of tensors in invariant subspaces. We replace in the standard tableaux 27 3 by 2 and 4 by 3. (The latter is obviously not essential). The result is

1 2 2 3	1 2 2 3	1 2 2 3	1 2 2 3	~~1~~ ~~2~~ ~~2~~ ~~3~~
	1 2 3 2	1 2 2 3	1 2 2 3	
	1 2 3 2		~~1 3~~ ~~2~~ ~~2~~	

The tableaux that I crossed out correspond to Young operators that annihilate $h_1h_2h_2h_3$. The normalized Young operators of those connected by brackets must be added to get the contributions to $h_1h_2h_2h_3$ from a particular subspace.

We do not know yet whether the invariant subspaces that we have found are irreducible. This will depend on the group $\mathcal{T}$. We will find in the next chapter for a special case and later generally that they are in fact irreducible for the $O(\infty)$ groups that we will consider, although these are only subgroups of the group of all bounded linear transformations in $L^2(\mathbb{R}_3)$.

I want to show in an example how the scalar product between two tensors, say $h_1'h_2'h_3'$ and $h_1h_2h_3$, splits into the contributions from various subspaces.

$$(h_1'h_2'h_3',h_1h_2h_3) = (h_1',h_1)(h_2',h_2)(h_3',h_3) \quad ,$$

where the quantities on the r.h.s. denote scalar products in $L^2(\mathbb{R}_3)$. First we write down how $h_1'h_2'h_3'$ and $h_1h_2h_3$ split:

sub- space	bra						ket					
	$h_1'h_2'h_3'$	$h_2'h_3'h_1'$	$h_3'h_1'h_2'$	$h_1'h_3'h_2'$	$h_2'h_1'h_3'$	$h_3'h_2'h_1'$	$h_1h_2h_3$	$h_2h_3h_1$	$h_3h_1h_2$	$h_1h_3h_2$	$h_2h_1h_3$	$h_3h_2h_1$
1 2 3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
1 2 3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
1 2 3	$\frac{1}{3}$		$-\frac{1}{3}$		$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$			$\frac{1}{3}$	$-\frac{1}{3}$
1 3 2	$\frac{1}{3}$	$-\frac{1}{3}$			$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		$-\frac{1}{3}$		$-\frac{1}{3}$	$\frac{1}{3}$

Now we give the contributions to the scalar product. These vanish whenever we take the contributions to the primed and unprimed quantities from subspaces corresponding to different tableaux. The nonvanishing scalar products are:

	$(h_1'h_1)(h_2'h_2)(h_3'h_3)$	$(h_1'h_2)(h_2'h_3)(h_3'h_1)$	$(h_1'h_3)(h_2'h_1)(h_3'h_2)$	$(h_1'h_1)(h_2'h_3)(h_3'h_2)$	$(h_1'h_3)(h_2'h_2)(h_3'h_1)$	$(h_1'h_2)(h_2'h_1)(h_3'h_3)$
$\begin{smallmatrix}1&2&3\end{smallmatrix}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$\begin{smallmatrix}1&2\\3\end{smallmatrix}$	$\frac{1}{3}$	$-\frac{1}{3}$			$-\frac{1}{3}$	$\frac{1}{3}$
$\begin{smallmatrix}1&3\\2\end{smallmatrix}$	$\frac{1}{3}$		$-\frac{1}{3}$		$\frac{1}{3}$	$-\frac{1}{3}$

5 A New Notation

For later applications I would like to write tensors in different invariant subspaces in such a way that we refer to corresponding tensors in equivalent representations with the same symbol. This is not the case in our notation. Take e.g. the scalar product between two tensors from $\begin{smallmatrix}1&2\\3\end{smallmatrix}$. It will in general not vanish. Replace the symbol for the second tensor by any for a tensor in the equivalent subspace $\begin{smallmatrix}1&3\\2\end{smallmatrix}$ and the expression will vanish.

$$\underline{Y}_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}h_1h_2h_3 = h_1h_2h_3-h_2h_3h_1+h_2h_1h_3-h_3h_2h_1 = A$$

$$\text{and}\quad \underline{Y}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}h_1h_3h_2 = h_1h_3h_2-h_2h_1h_3-h_3h_1h_2+h_2h_3h_1 = B$$

are corresponding tensors. To prove this, let [132] (within square brackets!) denote the operator that brings the h that stood in the 3^{rd} place to the 1^{st}, that from the 2^{nd} to the 3^{rd}, and that from the 1^{st} to the 2^{nd}:

$[132]B=A \ , \ U[T]A = U[T][132]B = [132]U[T]B \quad \forall T \ ,$

which proves our assertion. Similarly

$$\underline{Y}^{+}_{\substack{12\\3}} h_1' h_2' h_3' = A' \quad \text{and} \quad \underline{Y}^{+}_{\substack{13\\2}} h_1' h_3' h_2' = B'$$

are corresponding tensors since $-[123]B' = A'$.

Our new notation consists in the case of tensors in $\substack{13\\2}$ in replacing the old symbols B and B' by $[132]B$ and $-[123]B'$ respectively. Since tensors of the form B span $\substack{13\\2}$, this new prescription will always associate the same symbol with corresponding tensors in $\substack{12\\3}$ and $\substack{13\\2}$. It is obvious that our new notation suggests the same scalar product as before between two tensors in one and the same subspace.

Equal symbols in $\substack{12\\3}$ and $\substack{11\\2}$ denote corresponding tensors already in our old notation. In fact, $A = h_1h_2h_3 - h_2h_3h_1 + h_2h_1h_3 - h_3h_2h_1$ denotes not only the element $\underline{Y}_{\substack{12\\3}} h_1h_2h_3$ but also

$$\begin{aligned}\underline{Y}_{\substack{11\\2}} \tfrac{1}{3}\{&(h_1+h_2)(h_1+h_2)h_3 - h_1h_1h_3 - h_2h_2h_3 \\ &- (h_2+h_3)(h_2+h_3)h_1 + h_2h_2h_1 + h_3h_3h_1\}\end{aligned}$$

$$= \underline{Y}_{\substack{11\\2}} \tfrac{1}{3}\{h_1h_2h_3 + h_2h_1h_3 - h_2h_3h_1 - h_3h_2h_1\} \quad .$$

Since $U[T]A$ does not depend on whether we consider A as an element of $\substack{12\\3}$ or $\substack{11\\2}$, the above two elements will correspond to each other. The element in $\substack{12\\2}$ corresponding to these is

$$\underline{Y}_{\substack{12\\2}}\frac{1}{3}\{h_1h_2h_3+h_1h_3h_2-h_3h_1h_2-h_3h_2h_1\} = -[132]A \quad .$$

Similarly correspond to $\underline{Y}^{+}_{\substack{12\\3}}\ h'_1h'_2h'_3 = A' = -[123]B'$

besides $\underline{Y}^{+}_{\substack{13\\2}}\ h'_1h'_3h'_2 = B'$ also

$$\underline{Y}^{+}_{\substack{12\\2}}\ \frac{1}{2}\{-h'_3h'_1h'_2-h'_3h'_2h'_1\} = \frac{A'+B'}{2}$$

$$\text{and}\quad \underline{Y}^{+}_{\substack{11\\2}}\frac{1}{2}\{h'_1h'_2h'_3+h'_2h'_1h'_3\} \quad .$$

Such investigations may be cumbersome in more complicated cases. However, I think I have found a simple generally valid prescription for writing the contributions to a tensor from various invariant subspaces in such a way that the correct scalar product between two tensors in one and the same subspace is suggested and such that corresponding tensors are represented by the same symbol. I checked both properties in many cases that I considered likely to tell me if I was wrong but I did not give the general proof.

The prescription for writing the contribution to a given tensor from one of our invariant subspaces in the new notation can be symbolized by

$$X_\tau = \frac{1}{c_\tau}\underline{X}_\tau W_\tau \quad , \quad X^{+}_\tau = \frac{n_\tau}{c_\tau}\underline{X}^{+}_\tau W_\tau \quad . \tag{29}$$

X_τ and X^{+}_τ corresponding to a tableau τ containing n_1 numbers 1, n_2 numbers 2,..., and n_r numbers r are applicable to tensors that are symmetric in the first n_1, the second n_2,..., and the last n_r variables. Consider a tensor written in terms of the h's. Associate the i^{th} h from the left with the i^{th} lowest number in the tableau. W_τ tells you to reorder the h so that the corresponding numbers appear in the same order as in the tableau if we read its numbers like a text, i.e. from left to right beginning with the top line. $\underline{X}_\tau=\underline{\hat{P}}\,\underline{\hat{Q}}$, $\underline{X}_\tau^+=\underline{\hat{Q}}\,\underline{\hat{P}}$, where $\underline{\hat{P}}$ and $\underline{\hat{Q}}$ differ from $\underline{P}$ and $\underline{Q}$, that were introduced earlier, by putting [ ]-brackets around the number sequences instead of the ( ) ones. The meaning of the square brackets has been explained above. Note that P and Q appear in X and Y in opposite order. c_τ was introduced before: $c_\tau\underline{Y}_\tau=\underline{Y}_\tau^2$, but also $c_\tau\underline{X}_\tau=\underline{X}_\tau^2$.

$$n_\tau = \frac{\prod_j (n_j)!}{\prod_{j,i}(n_{ij})!} \quad , \tag{30}$$

where n_j gives the number of j's in the tableau, n_{ij} the number of j's in the i^{th} row.

This rule is much simpler than it sounds. Let me illustrate it for $\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2 \\\cline{1-1}\end{array}$ and $h_4h_2h_2h_2$. W_τ has in this case no effect since the numbers in the tableau are already in ascending order.

$$\underline{X}_\tau = (e+[12]+[23]+[31]+[123]+[132])(e-[14]) \quad ,$$

$$c_\tau = \frac{4!}{3} = 8 \quad , \quad n_\tau = \frac{3!}{2!} = 3$$

Hence $X_\tau h_4h_2h_2h_2 = \frac{1}{8}\underline{X}_\tau h_4h_2h_2h_2$

$$= \frac{1}{8}(-6h_2h_2h_2h_4+2h_2h_2h_4h_2+2h_2h_4h_2h_2+2h_4h_2h_2h_2)$$

$$X_\tau^+ h_4h_2h_2h_2 = \frac{3}{8}\underline{X}_\tau^+ h_4h_2h_2h_2$$

$$= \frac{3}{8}(-2h_2h_2h_2h_4 \qquad\qquad\qquad +2h_4h_2h_2h_2)$$

We will apply this prescription more extensively in later chapters but obviously not in situations where its correctness would be crucial.

V ADDITIONAL FEATURES OF ROTATIONALLY-SYMMETRIC MODELS

In this chapter I shall apply the methods developped in the last chapter to RS-models. The group $\mathcal{G}$ is here $O(\infty,\mathbb{R}_3)$; the quantities $h_1, h_2, h_3, \ldots$ are now restricted to two, which I shall call h and $\hbar$. I will derive the general form of the matrix elements of the operators that commute with the $U[T]$ $\forall\, T\varepsilon O(\infty,\mathbb{R}_3)$, then investigate what this tells me about $\mathcal{H}$, which is a particular such operator. Finally I shall introduce creation and absorption operators, which will shed light on the previous results from a different and very interesting angle.

1 Exponential Hilbert Space

From theorem 1 in chapter III it can easily be derived using some of the assumptions (i) and (ii) that the overlap between two arbitrary members of our overcomplete family of states is given by

$$(\Phi[f',g'],\Phi[f,g])$$
$$= e^{-\frac{i}{2}\{(f',g)-(g',f)\}}\, e^{-\frac{1}{4}\{\frac{\xi}{m}(f-f',f-f')+m(g-g',g-g')\}} .$$

Introducing

$$h = \frac{f-img}{\sqrt{2m}} \quad \text{and} \quad \hbar = \frac{\eta f}{\sqrt{2m}} \qquad 31$$

with

$$\eta = \sqrt{\xi-1}$$

we can write this as

$$(\Phi[f',g'],\Phi[f,g]) = N'N \;\; e^{(h',h)+(\hbar',\hbar)} , \qquad 32$$

where N is short for

$$N_{\Phi[f,g]} = (\Phi_0, \Phi[f,g]) \quad .$$

Eq. 32 tells us that we can write the overlap as a sum

$$(\Phi[f',g'], \Phi[f,g]) = \sum_n \frac{N'N}{n!} \{(h',h) + (\hbar',\hbar)\}^n \quad ,$$

whose terms we can consider as describing the contributions to the scalar product from the subspaces $\mathcal{H}_n$ of $\mathcal{H}$.

$$\mathcal{H} = \sum_n \oplus \mathcal{H}_n \qquad 33$$

We will sometimes refer to the $\mathcal{H}_n$ as the sectors of $\mathcal{H}$. They are symmetrized direct products of n factors h.

$$h = L^2(\mathbb{R}_3) \oplus L^2(\mathbb{R}_3)$$

in the case of reducible representations of the CCR and

$$h = L^2(\mathbb{R}_3)$$

in the case of irreducible ones since $\hbar$ vanishes then. $\mathcal{H}_0$ is one-dimensional and contains the vacuum Φ_0.

The contributions $\Phi_n[f,g]$ to $\Phi[f,g]$ from the sectors $\mathcal{H}_n$ are given by

n	$\Phi_n[f,g]$
0	N
1	$N \frac{h(\underline{x}) \oplus \hbar(\underline{x})}{\sqrt{1!}}$
2	$N \frac{(h(\underline{x}_1) \oplus \hbar(\underline{x}_1))(h(x_2) \oplus \hbar(x_2))}{\sqrt{2!}}$
.	
.	
n	$N \frac{(h(\underline{x}_1) \oplus \hbar(\underline{x}_1)) \ldots (h(\underline{x}_n) \oplus \hbar(\underline{x}_n))}{\sqrt{n!}}$
.	
.	

2 Weak Convergence in $\hbar$ and in $\mathcal{H}$

We abbreviate elements $h(\underline{x})\oplus\hbar(\underline{x})$ by small Greek letters and the corresponding $\Phi[f,g]$ by the corresponding Greek capitals. The scalar product of two elements $\phi = h'(\underline{x})\oplus\hbar'(\underline{x})$ and $\psi = h(\underline{x})\oplus\hbar(\underline{x})$ of $\hbar$ is defined as $(h',h)\oplus(\hbar',\hbar)$ and will be denoted by (ϕ,ψ). Assume $\psi_n \rightharpoonup \psi$, i.e. $\lim_{n\to\infty} (\lambda,\psi_n) = (\lambda,\psi)\ \forall\lambda\in\hbar$. I want to show that

$$\psi_n \rightharpoonup \psi \quad \text{implies} \quad \frac{\Psi_n}{N_{\Psi_n}} \rightharpoonup \frac{\Psi}{N_\Psi} \quad . \tag{34}$$

We need only prove $\lim_{n\to\infty} \frac{(\Lambda,\Psi_n)}{N_{\Psi_n}} = \frac{(\Lambda,\Psi)}{N_\Psi}$ since our cyclicity assumption says that the Λ form a complete set:

$$\lim_{n\to\infty} \frac{(\Lambda,\Psi_n)}{N_{\Psi_n}} = \lim_{n\to\infty} N_\Lambda e^{(\lambda,\psi_n)} = N_\Lambda e^{(\lambda,\psi)} = \frac{(\Lambda,\Psi)}{N_\Psi} \quad .$$

3 $\mathcal{H}$ Viewed as a Representation Space for $O(\infty,\mathbb{R}_3)$

$$(\Phi[f',g'],U[T]\Phi[f,g]) = \sum_n \frac{N'N}{n!}\{(h',Th)+(\hbar',T\hbar)\}^n$$

tells us that each $\mathcal{H}_n$, $n=0,1,2,\ldots$, is invariant under the $U[T]$ $\forall T\in O(\infty,\mathbb{R}_3)$. We can therefore write

$$U[T] = \sum_{n=0}^{\infty} \oplus\, U_n[T] \quad ,$$

where $U_n[T]$ is the operator $U[T]$ restricted to $\mathcal{H}_n$, i.e. replaced by 0 outside $\mathcal{H}_n$. We want to show that the various $\mathcal{H}_n$ are disjoint, by which we mean that no representation contained in any $\mathcal{H}_n$ is equivalent to a representation in any other $\mathcal{H}_p$. In formal language we want to show the impossibility of having

$$QU_n[T]Q = VPU_p[T]PV^{-1}$$

$\forall\, T\epsilon O(\infty,\mathbb{R}_3)$, where Q and P are projection operators and V is unitary. The proof is due to Klauder (reference 3), and I will reproduce it here since I will be able to exploit it more fully than was done hitherto. We consider a sequence $T_m \to cI$, $T_m \epsilon O(\infty,\mathbb{R}_3)$ for $m=1,2,\ldots$. Such sequences exist: We introduce into $L^2(\mathbb{R}_3)$ a basis as in the previous chapter and choose

$$T_m = \left[\begin{array}{c|c} \left.\begin{array}{cc} \begin{matrix} c & & \\ & \ddots & \\ & & c \end{matrix} & \begin{matrix} -s & & \\ & \ddots & \\ & & -s \end{matrix} \\ \begin{matrix} s & & \\ & \ddots & \\ & & s \end{matrix} & \begin{matrix} c & & \\ & \ddots & \\ & & c \end{matrix} \end{array}\right\} 2m & \\ \hline & \begin{array}{cc} \begin{matrix} c & & \\ & \ddots & \\ & & c \end{matrix} & \begin{matrix} -s & \\ & \ddots \end{matrix} \\ \begin{matrix} s & \\ & \ddots \end{matrix} & \begin{matrix} c & \\ & \ddots \end{matrix} \end{array} \end{array}\right] \qquad 35$$

where c stands for cos Θ, s for sin Θ.

$$\lim_{m\to\infty} (\Phi[f',g'],U[T_m]\Phi[f,g]) = \lim_{m\to\infty} (\Phi[f',g'],\Phi[T_mf,T_mg])$$

$$= (\Phi[f',g'],\frac{N_{\Phi[f,g]}}{N_{\Phi[cf,cg]}}\Phi[cf,cg])$$

Because the $\|U[T_m]\|$ are uniformly bounded (they are in fact all 1) and because the above equation holds $\forall\, \Phi[f',g']$ and $\Phi[f,g]$, the silver rule (Cf. appendix A) ensures us that

$$U[T_n] \to \frac{N_{\Phi[f,g]}}{N_{\Phi[cf,cg]}} \sum_{n=0}^{\infty} \oplus\, c^n I_n \quad ,$$

where I_n is the identity operator in $\mathcal{H}_n$. However,

$$Qc^n I_n Q = VPc^p I_p PV^{-1}$$

can, unless $n=p$, only hold $\forall$ c between -1 and +1 if $Q=P=0$.

4 Operators Commuting with all the U[T]

We consider matrix elements of an arbitrary bounded operator $\mathcal{B}$ that commutes with $U[T]$ $\forall$ $T\epsilon O(\infty,\mathbb{R}_3)$. We call the set of these operators the commutator of the U[T] and write it as $\{U[O(\infty,\mathbb{R}_3)]\}'$ or shorter as $\{U[T]\}'$.

$$\begin{aligned}(\Phi[f',g'],\mathcal{B}\Phi[f,g]) &= (\Phi[f',g'],U^{-1}[T]\mathcal{B}U[T]\,\Phi[f,g]) \\ &= (\Phi[Tf',Tg'],\mathcal{B}\Phi[Tf,Tg])\end{aligned}$$

$\forall$ $T\epsilon O(\infty,\mathbb{R}_3)$. These matrix elements can therefore only depend on

$(f;f'),(f;g'),(g;g'),(f;f),(f;g),(g;f),(g;g),(f,f),(f,g),(g,g)$

or equivalently on

$(h;h'),(h;\hbar'),(\hbar;\hbar'),(h;h),(h;\hbar),(\hbar;h),(\hbar;\hbar),(h,h),(h,\hbar),(\hbar,\hbar)$.

Disjointness of $\mathcal{H}$ as a representation space for the U[T] tells us that $\mathcal{B}\epsilon\{U[T]\}'$ must have the structure

$$\mathcal{B} = \sum_{n=0}^{\infty} \oplus \mathcal{B}_n \quad , \qquad 36$$

where $\mathcal{B}_n$ is the restriction of $\mathcal{B}$ to $\mathcal{H}_n$. We can therefore put

$$(\Phi_n[f',g'],\mathcal{B}_n\Phi_n[f,g])$$

$$= N_{\Phi[f;g']}N_{\Phi[f,g]}A_n[(h;h'),(h;\hbar'),(\hbar;\hbar'),(h;h),(h;\hbar),(\hbar;h), (\hbar;\hbar),(h,h),(h,\hbar),(\hbar,\hbar)] \quad .$$

Hence

$$\lim_{m\to\infty} \left(\Phi_n[f',g'],\mathcal{B}_n\Phi_n[T_mf,T_mg]\right)$$

$$= \lim_{m\to\infty} N_\Phi[f;g']\underbrace{N_\Phi[T_mf,T_mg]}_{N_\Phi[f,g]}A_n[(h;h'),\ldots,(h;T_mh),\ldots,\underbrace{(T_mh,T_mh)}_{(h,h)},.]$$

$$= N_\Phi[f;g']N_\Phi[f,g]A_n[(h;h'),\ldots,c(h;h),\ldots,(h,h),\ldots] \,. \qquad 37$$

On the other hand we have that

$$\lim_{m\to\infty} (\Phi[f;g'],\mathcal{B}\Phi[T_mf,T_mg]) = \frac{N_\Phi[f,g]}{N_\Phi[cf,cg]}(\Phi[f;g'],\mathcal{B}\Phi[cf,cg]) \quad ,$$

to which the n'th sector contributes

$$(\Phi_n[f;g'],\mathcal{B}_n\Phi_n[cf,cg])$$

$$= \frac{N_\Phi[f,g]}{\cancel{N_\Phi[cf,cg]}}N_\Phi[f;g']\cancel{N_\Phi[cf,cg]}A_n[(h;h'),\ldots,c(h;h),\ldots,c^2(h,h),..]$$

Comparing this with eq. 37 we find that A_n cannot depend on $(h,h),(h,\hbar),(\hbar,\hbar)$. Similarly one sees that it also cannot depend on $(h;h'),(h;\hbar'),(\hbar;\hbar')$. Since

$$\frac{\Phi_n[cf,cg]}{N_\Phi[cf,cg]} = c^n\,\frac{\Phi[f,g]}{N_\Phi[f,g]} \quad ,$$

it follows that A_n is homogeneous of degree n. It must be a polynomial in its arguments. Assume this were not the case, introduce a basis in $L^2(\mathbb{R}_3)$, and consider a d-dimensional subspace of the ensuing l^2 space. A_n will not reduce to a polynomial in this subspace. $\mathcal{B}$ can therefore not lie in $\{U[O(d)]\}'$ and a fortiori not in $\{U[O(\infty,\mathbb{R}_3)]\}'$. It follows that we can write

$$(\Phi[f;g'],\mathcal{B}\Phi[f,g])$$

$$=N'N\sum_{p,q=0}^{\infty}\frac{1}{p!q!}\sum_{r=0}^{p}\sum_{s=0}^{q} b_{pr,qs}(h;h)^{p-r}(h;\hbar)^{r}(\hbar'h)^{s}(\hbar'\hbar)^{q-s} \quad . \qquad 38$$

To $\mathcal{B}=I$ corresponds

$$b_{pr,qs} = \delta_{r0}\delta_{s0} \quad .$$

In order that the $b_{pr,qs}$ determine in fact a bounded operator $\mathcal{B}$, they must be sufficiently well behaved for large values of their indices. We saw, eq. 36, that $\mathcal{B}=\sum_{n=0}^{\infty}\oplus\mathcal{B}_n$. Different sequences $\mathcal{B}_n$ correspond to different $\mathcal{B}$. The

$$\sum_{q=0}^{n}(n-q+1)(q+1) = \frac{(n+1)(n+2)(n+3)}{3!}$$

complex numbers $b_{pr,qs}$ with $p+q=n$ determine $\mathcal{B}_n$ completely. Different such sets correspond to different $\mathcal{B}_n$ in the case of reducible representations of the CCR, whereas in the case of irreducible representations we can choose

$$b_{n-q\ r,q\ s} = b_n\ \delta_{q0}\ \delta_{r0}\ \delta_{s0}.$$

5 Irreducible Representations of the CCR

In this case eq. 38 reduces to

$$(\Phi[f;g'],\mathcal{B}\Phi[f,g]) = N'N\sum_{n=0}^{\infty}\frac{1}{n!}\,b_n(h;h)^n \quad . \qquad 39$$

On the other hand we know that each $\mathcal{H}_n$ is invariant under $U[O(\infty,\mathbb{R}_3)]$. Assume for the moment that they are also irreducible. Schur's lemma tells us then, since the $\mathcal{H}_n$ are disjoint, that each $\mathcal{B}_n$ must be a multiple of the identity, say $b_n \cdot I$. They will correspond to $\mathcal{B}\epsilon\{U[O(\infty,\mathbb{R}_3)]\}'$ if and only if the b_n are uniformly bounded. The matrix elements of this operator are obviously just those given by eq. 39. The $\mathcal{H}_n$ are therefore in fact irreducible since otherwise there would be operators $\mathcal{B}$ whose matrix elements are not of the form given in eq. 39. We introduce a new notation

$$(\Phi[f';g'],\mathcal{B}\Phi[f,g])$$
$$=\left(N';N'h'^*(\underline{x}),N'\frac{h'^*(\underline{x}_1)h'^*(\underline{x}_2)}{\sqrt{2!}},\ldots\right)\begin{pmatrix} b_0 & 0 & 0 & \\ 0 & b_1 & 0 & \\ & & & \\ 0 & 0 & b_2 & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} N \\ Nh(\underline{x}) \\ N\frac{h(\underline{x}_1)h(\underline{x}_2)}{\sqrt{2!}} \\ \vdots \end{pmatrix}.$$

Comparing with eq. 39 we note that on the r.h.s. not only matrix multiplication is implied but also integration over arguments that appear twice.

This notation lies halfway between describing elements of $\mathcal{H}$ just by a symbol and giving the coefficients with respect to a basis. Each entry in the column vector is not just a coefficient of $\Phi[f,g]$ but the contribution to $\Phi[f,g]$ from a subspace that is irreducible under the $U[T]$. This notation is useful if one considers only operators that commute with the $U[T]$. They are represented by matrices. The elegance of the notation will become clear in the case of reducible representations of the CCR, where the matrices need no longer be diagonal because of the appearance of equivalent representations.

6 Reducible Representations of the CCR

In this case the $\mathcal{H}_n$ are no longer irreducible except for $\mathcal{H}_0$. $\mathcal{H}_1$ is the direct sum of two invariant subspaces, $\mathcal{H}_1 = \mathcal{H}^{10} \oplus \mathcal{H}^{01}$, where $\mathcal{H}^{10}$ contributes $Nh(\underline{x})$ and $\mathcal{H}^{01}$ $N\hbar(\underline{x})$ to $\Phi[f,g]$. Both subspaces are irreducible since the representations that they carry are both equivalent to the representation in $\mathcal{H}_1$ in the case of irreducible representations of the CCR. In general, it is clear that $\mathcal{H}_n$ is reducible into invariant subspaces $\mathcal{H}^{p,q}$ with $p=0,1,\dots,n$; $q=n-p$. The contribution to $\Phi[f,g]$ from $\mathcal{H}^{p,q}$ is $\frac{N}{\sqrt{p!q!}}\left(\prod_{i=1}^{p} h(\underline{x}_i)\right)\left(\prod_{i=p+1}^{n} \hbar(\underline{x}_i)\right)$.

n	$\Phi_n[f,g]$
0	N
1	$\frac{N}{\sqrt{1!}}\left(h(\underline{x})\oplus\hbar(\underline{x})\right)$
2	$\frac{N}{\sqrt{2!}}\left(h(\underline{x}_1)\oplus\hbar(\underline{x}_1)\right)\left(h(\underline{x}_2)\oplus\hbar(\underline{x}_2)\right)$
	$= N\left[\frac{h(\underline{x}_1)h(\underline{x}_2)}{\sqrt{2!}}\oplus h(\underline{x}_1)\hbar(\underline{x}_2)\oplus\frac{\hbar(\underline{x}_1)\hbar(\underline{x}_2)}{\sqrt{2!}}\right]$
3	$\frac{N}{\sqrt{3!}}(h\oplus\hbar)(h\oplus\hbar)(h\oplus\hbar)$
.	$= N\left[\frac{hhh}{\sqrt{3!}}\oplus\frac{hh\hbar}{\sqrt{2!}}\oplus\frac{h\hbar\hbar}{\sqrt{2!}}\oplus\frac{\hbar\hbar\hbar}{\sqrt{3!}}\right]$
.	
.	
n	$\frac{N}{\sqrt{n!}}\prod_{i=1}^{n}\left(h(\underline{x}_i)\oplus\hbar(\underline{x}_i)\right)$
.	
.	$= N\sum_{p=n,n-1,\dots,1}^{\oplus}\frac{h(\underline{x}_1)\dots h(\underline{x}_p)\hbar(\underline{x}_{p+1})\dots\hbar(\underline{x}_n)}{\sqrt{p!}\,\sqrt{(n-p)!}}$
.	

We know from the previous chapter that the representations in $\mathcal{H}^{p,q}$ and $\mathcal{H}^{q,p}$ are equivalent but in general not irreducible. For the lowest sectors of Hilbert space, I shall list below the invariant subspaces which we already know of, and I shall give the contributions to $\Phi[f,g]$ that they contain. I need not give them for kets and bras separately since they look the same. This is due to the fact that to each graph there corresponds only one standard tableau containing a given number of ones and the rest twos. I shall list also, for kets and bras, the symbols introduced at the end of the last chapter.

Sub-space		Contribution in: original notation	new notation: kets	new notation: bras
		N		
$\mathcal{H}^{10}$	1	$Nh(\underline{x})$		
$\mathcal{H}^{01}$	2	$N\hbar(\underline{x})$		
$\mathcal{H}^{20}$	11	$N\frac{h(\underline{x}_1)h(\underline{x}_2)}{\sqrt{2!}}$		
$\mathcal{H}^{11}$	12	$N\frac{h(\underline{x}_1)\hbar(\underline{x}_2)+\hbar(\underline{x}_1)h(\underline{x}_2)}{2}$		
	$\frac{1}{2}$	$N\frac{h(\underline{x}_1)\hbar(\underline{x}_2)-\hbar(\underline{x}_1)h(\underline{x}_2)}{2}$		
$\mathcal{H}^{02}$	22	$N\frac{\hbar(\underline{x}_1)\hbar(\underline{x}_2)}{\sqrt{2!}}$		
$\mathcal{H}^{30}$	111	$N\frac{hhh}{\sqrt{3!}}$		
$\mathcal{H}^{21}$	112	$N\frac{hh\hbar+h\hbar h+\hbar hh}{3\sqrt{2!}}$		
	$\frac{1}{2}1$	$N\frac{2hh\hbar-h\hbar h-\hbar hh}{3\sqrt{2!}}$		$N\frac{2hh\hbar-2\hbar hh}{3\sqrt{2!}}$
$\mathcal{H}^{12}$	122	$N\frac{h\hbar\hbar+\hbar h\hbar+\hbar\hbar h}{3\sqrt{2!}}$		
	$\frac{1}{2}2$	$N\frac{2h\hbar\hbar-\hbar h\hbar-\hbar\hbar h}{3\sqrt{2!}}$	1	$N\frac{2h\hbar\hbar-2\hbar\hbar h}{3\sqrt{2!}}$
$\mathcal{H}^{03}$	222	$N\frac{\hbar\hbar\hbar}{\sqrt{3!}}$		
$\mathcal{H}^{40}$	1111	$N\frac{hhhh}{\sqrt{4!}}$		
$\mathcal{H}^{31}$	1112	$N\frac{hhh\hbar+hh\hbar h+h\hbar hh+\hbar hhh}{4\sqrt{3!}}$		
	$\frac{1}{2}11$	$N\frac{3hhh\hbar-hh\hbar h-h\hbar hh-\hbar hhh}{4\sqrt{3!}}$		$N\frac{3hhh\hbar-3\hbar hhh}{4\sqrt{3!}}$
$\mathcal{H}^{22}$	1122	$N\frac{hh\hbar\hbar+h\hbar h\hbar+h\hbar\hbar h+\hbar hh\hbar+\hbar h\hbar h+\hbar\hbar hh}{6(2!)}$		
	$\frac{1}{2}12$	$N\frac{3hh\hbar\hbar-3\hbar\hbar hh}{6(2!)}$	2	$N\frac{2hh\hbar\hbar+2h\hbar h\hbar-2\hbar h\hbar h-2\hbar\hbar hh}{4(2!)}$
	$\frac{1}{2}\frac{1}{2}$	$N\frac{2hh\hbar\hbar-h\hbar h\hbar-h\hbar\hbar h-\hbar hh\hbar-\hbar h\hbar h+2\hbar\hbar hh}{6(2!)}$		$N\frac{2hh\hbar\hbar-2h\hbar h\hbar-2\hbar h\hbar h+2\hbar\hbar hh}{6(2!)}$
$\mathcal{H}^{13}$	1222	$N\frac{h\hbar\hbar\hbar+\hbar h\hbar\hbar+\hbar\hbar h\hbar+\hbar\hbar\hbar h}{4\sqrt{3!}}$		
	$\frac{1}{2}22$	$N\frac{3h\hbar\hbar\hbar-\hbar h\hbar\hbar-\hbar\hbar h\hbar-\hbar\hbar\hbar h}{4\sqrt{3!}}$	3	$N\frac{3h\hbar\hbar\hbar-3\hbar\hbar\hbar h}{4\sqrt{3!}}$
$\mathcal{H}^{04}$	2222	$N\frac{\hbar\hbar\hbar\hbar}{\sqrt{4!}}$		

Remarks

The expression can be taken over from the left where I left a place empty. The new notation differs from the original one for kets in three cases only:

1 $N\frac{h\hbar\hbar+\hbar h\hbar-2\hbar\hbar h}{3\sqrt{2!}}$

2 $N\frac{hh\hbar\hbar+h\hbar h\hbar-h\hbar\hbar h+\hbar hh\hbar-\hbar h\hbar h-\hbar\hbar hh}{4(2!)}$

3 $N\frac{h\hbar\hbar\hbar+\hbar h\hbar\hbar+\hbar\hbar h\hbar-3\hbar\hbar\hbar h}{4\sqrt{3!}}$

It is easy to see that the invariant subspaces listed in the table must be irreducible. Assume for the moment that this is true. Introduce as in the case of irreducible CCR representations a matrix notation where each entry corresponds to an irreducible subspace and use the new notation. $U[T]$ will then for each fixed T look the same in equivalent subspaces since one and the same symbol denotes corresponding vectors in them. Each $\mathcal{B}\in\{U[O(\infty,\mathbb{R}_3)]\}'$ will be represented by a matrix with constant coefficients that has nonzero elements only between equivalent subspaces. (If we used the original notation, $\mathcal{B}_n$ could not be characterized so simply.) The number of independent elements in the matrix characterizing $\mathcal{B}_n$ equals the sum of the squares of the numbers of equivalent representations in $\mathcal{H}_n$, which is

$$(n+1)^2+(n-1)^2+(n-3)^2+\dots+\genfrac{}{}{0pt}{}{2^2}{\text{or }1^2} = \frac{(n+1)(n+2)(n+3)}{3!} \quad .$$

On the other hand we saw earlier that the $b_{pr,qs}$, too, determine $\mathcal{B}$ completely and that there are just $\frac{(n+1)(n+2)(n+3)}{3!}$ $b_{pr,qs}$'s with $p+q=n$. $\mathcal{H}_n$ can therefore not be further reducible.

The matrix describing $\mathcal{B}$ will thus look as follows

41

All matrix elements outside the blackened areas will vanish.

For a given such matrix check for each blackened square whether it can be replaced by a number of smaller ones along the main diagonal in such a way that every matrix element outside vanishes. Take the sum of the absolute squares of the matrix elements in each of these minimal squares. A sufficient condition for the matrix to correspond to a bounded operator $\mathcal{B}$ is that these sums be uniformly bounded.

The Hamiltonian $\mathcal{H}$ is a self-adjoint operator that commutes with all the $U[T]$. We can therefore write it in the above matrix form, too. The restriction on the magnitude of the matrix elements that we have for operators in $\{U[O(\infty,\mathbb{R}_3)]\}'$ does not apply to $\mathcal{H}$ because it is unbounded. The matrix describing $\mathcal{H}$ will be symmetric.

7 Creation and Annihilation Operators for Irreducible Representations

Introduction of the operators

I claim that

$$A(\alpha)\prod_{i=1}^{n} h(\underline{x}_i) = \sqrt{n}(\alpha,h)\prod_{i=1}^{n-1} h(\underline{x}_i) \qquad 42$$

for $n = 1,2,\ldots$ and $A(\alpha)\Phi_0 = 0$

defines for each $\alpha\in L^2(\mathbb{R}_3)$ a closable linear operator $A(\alpha)$ acting on $\mathcal{H}$. 42 shows that $A(\alpha)$ is linear and that it maps every element of $\mathcal{H}_n$ into an element of $\mathcal{H}_{n-1}$. It follows from 42 how $A(\alpha)$ acts on $\Phi[f,g]$:

$$A(\alpha)\Phi[f,g] = A(\alpha)N\sum_{n=0}^{\infty}\oplus\frac{1}{\sqrt{n!}}\prod_{i=1}^{n} h(\underline{x}_i) = (\alpha,h)\Phi[f,g] \quad . \qquad 43$$

Linearity of $A(\alpha)$ allows us to extend the definition of $A(\alpha)$ to linear combinations of $\Phi[f,g]$, which we know to be dense in $\mathcal{H}$. $A(\alpha)$ is therefore a linear operator. In order to prove that it is closable we show that it possesses an adjoint $A^+(\alpha)$.

$$A^+(\alpha)\prod_{i=1}^{n}h(\underline{x}_i) = \frac{1}{\sqrt{n+1}}\cdot \tag{42'}$$

$$\{\alpha(\underline{x}_1)h(\underline{x}_2)..h(\underline{x}_{n+1})+h(\underline{x}_1)\alpha(\underline{x}_2)..h(\underline{x}_{n+1})+..+h(\underline{x}_1)..h(\underline{x}_n)\alpha(\underline{x}_{n+1})\}$$

This relation simplifies for the special case $\alpha=h=\frac{f-img}{\sqrt{2m}}$ to

$$A^+(h)\prod_{i=1}^{n}h(\underline{x}_i) = \sqrt{n+1}\prod_{i=1}^{n+1}h(\underline{x}_i) \quad . \tag{42''}$$

It follows from 42' that

$$\begin{aligned} A^+(\alpha)\Phi[f,g] &= A^+(\alpha)N\sum_{n=0}^{\infty}\oplus\frac{1}{\sqrt{n!}}\prod_{i=1}^{n}h(\underline{x}_i) \\ &= N\sum_{n=0}^{\infty}\oplus\frac{\alpha(\underline{x}_1)h(\underline{x}_2)...h(\underline{x}_{n+1})+...+h(\underline{x}_1)...h(\underline{x}_n)\alpha(\underline{x}_{n+1})}{\sqrt{(n+1)!}} \quad . \end{aligned} \tag{43'}$$

$A^+(\alpha)$ is a linear operator. Let us prove that it is in fact the adjoint of $A(\alpha)$. We have to show:

$$\begin{aligned} (\Phi[f';g'],A^+(\alpha)\Phi[f,g]) &= (\Phi[f,g],A(\alpha)\Phi[f';g'])^* \\ &= (h';\alpha)(\Phi[f';g'],\Phi[f,g]) \quad . \end{aligned}$$

In fact

$$\begin{aligned} (\Phi[f';g'],A^+(\alpha)\Phi[f,g]) &= N'N\sum_{n=0}^{\infty}\frac{(n+1)(h';\alpha)(h';h)^n}{(n+1)!} \\ &= (h';\alpha)(\Phi[f';g'],\Phi[f,g]) \quad . \end{aligned}$$

From the definition of A and A^+ it follows immediately that

$$[A(\alpha),A(\beta)] = [A^+(\alpha),A^+(\beta)] = 0 \tag{44}$$

and evaluating $[A(\alpha),A^+(\beta)]\prod_{i=1}^{n}h(\underline{x}_i)$ one finds

$$[A(\alpha),A^+(\beta)] = (\alpha,\beta) \quad . \tag{45}$$

Equations 44 and 45 are the commutation relations of absorption and creation operators. We shall call $A(\alpha)$ an absorption, $A^+(\alpha)$ a creation operator and $\mathcal{H}_n$ the n-particle sector of Hilbert space.

We can write

$$A(\alpha) = \int d\underline{x} \quad \alpha^*(\underline{x})A(\underline{x}) \quad .$$

$A(\underline{x})$ is a linear operator and it has been used by Klauder in his lectures at Brandeis (reference 3). I prefer to deal with the smeared operators since $A(\underline{x})$ is not closable and does therefore not possess an adjoint. If one tries to define a quantity $A^+(\underline{x})$ at least on a subspace of $\mathcal{H}$, one easily convinces oneself that this is possible for the null-vector only, not for Φ_o.

The Hamiltonian

Let e_n, $n=1,2,...$ denote an orthonormal basis in $L^2(\mathbb{R}_3)$. We claim that the Hamiltonian can be written as

$$\mathcal{H} = m\sum_n A^+(e_n)A(e_n) \quad .$$

Using $\sum_n e_n(\underline{x}_i)(e_n,h) = h(\underline{x}_i)$, we find

$$\mathcal{H}\prod_{i=1}^{n} h(\underline{x}_i) = n\cdot m\prod_{i=1}^{n} h(\underline{x}_i) \quad .$$

$\mathcal{H}$ acts thus in each irreducible subspace $\mathcal{H}_n$ of $\mathcal{H}$ as a multiple of the identity, namely $n\cdot m\cdot I$. It is therefore an operator commuting with the $U[T]$ $\forall$ $T\in O(\infty,\mathbb{R}_3)$ but obviously not a bounded one. Its elements between states of our overcomplete family are

$$(\Phi[f';g'],H\Phi[f,g]) = \frac{1}{2}(f'-img',f-img)(\Phi[f';g'],\Phi[f,g])$$

in agreement with eq. 20.

$\Phi_n[f,g]$ and $\Phi[f,g]$ expressed in terms of $A^+(h)$

It follows from eq. 42" that

$$\Phi_n[f,g] = N\frac{(A^+(h))^n}{n!}\Phi_o \quad , \qquad 46$$

and hence

$$\Phi[f,g] = Ne^{A^+(h)}\Phi_o \quad . \qquad 47$$

8 Creation and Annihilation Operators for Reducible Representations

Introduction of the operators

Define operators $A(\alpha_1\oplus\alpha_2)$ by

$$A(\alpha_1\oplus\alpha_2)(\prod_{i=1}^{p}h(\underline{x}_i)\prod_{i=p+1}^{n}\hbar(\underline{x}_i))$$

$$= \sqrt{p}(\alpha_1,h)\prod_{i=1}^{p-1}h(\underline{x}_i)\prod_{i=p}^{n-1}\hbar(\underline{x}_i)\oplus\sqrt{q}(\alpha_2,\hbar)\prod_{i=1}^{p}h(\underline{x}_i)\prod_{i=p+1}^{n-1}\hbar(\underline{x}_i) \quad , \qquad 48$$

where q stands for n-p. $A(\alpha_1\oplus\alpha_2)$ maps an element of $\mathcal{H}^{p,q}$ into an element of $\mathcal{H}^{p-1,q}\oplus\mathcal{H}^{p,q-1}$. The adjoint of $A(\alpha_1\oplus\alpha_2)$ satisfies $A^+(\alpha_1\oplus\alpha_2)(\prod_{i=1}^{p}h(\underline{x}_i)\prod_{i=p+1}^{n}\hbar(\underline{x}_i))$

$$= \frac{\alpha_1(\underline{x}_1)h(\underline{x}_2)..h(\underline{x}_{p+1})+..+h(\underline{x}_1)..h(\underline{x}_p)\alpha_1(\underline{x}_{p+1})}{\sqrt{p+1}}(\prod_{i=p+2}^{n+1}\hbar(\underline{x}_i))$$

$$\oplus \quad (\prod_{i=1}^{p}h(\underline{x}_i))\frac{\alpha_2(\underline{x}_{p+1})\hbar(\underline{x}_{p+2})..\hbar(\underline{x}_{n+1})+..+\hbar(\underline{x}_{p+1})..\hbar(\underline{x}_n)\alpha(\underline{x}_{n+1})}{\sqrt{q+1}} \quad . \qquad 49$$

Again we may consider $A(\alpha_1\oplus\alpha_2)$ as the result of smearing the non-closable operator $A(\underline{x})$ with $\alpha_1\oplus\alpha_2\in\mathcal{h}$,

$$A(\alpha_1\oplus\alpha_2) = \int d\underline{x}\ (\alpha_1{}^*(\underline{x})\oplus\alpha_2{}^*(\underline{x}))A(\underline{x}) \quad .$$

Consider the special cases

$$A_1(\alpha) = A(\alpha \oplus 0) \qquad 50$$

and $$A'(\alpha) = A\left(\frac{i\eta}{\sqrt{1+\eta^2}}\alpha \oplus \frac{-i}{\sqrt{1+\eta^2}}\alpha\right) = \frac{-i}{\sqrt{1+\eta^2}} A(\eta\alpha \oplus -\alpha) \quad , \qquad 51$$

which will turn out to be convenient for a discussion of the Hamiltonian. These operators obey the commutation relations

$$\begin{aligned}
&[A_1(\alpha),A_1(\beta)] = [A_1^+(\alpha),A_1^+(\beta)] = 0 \quad , \quad [A_1(\alpha),A_1^+(\beta)] = (\alpha,\beta) \quad ; \\
&[A'(\alpha),A'(\beta)] = [A'^+(\alpha),A'^+(\beta)] = 0 \quad , \quad [A'(\alpha),A'^+(\beta)] = (\alpha,\beta) \quad ; \\
&[A_1(\alpha),A'(\beta)] = [A_1^+(\alpha),A'^+(\beta)] = 0 \quad , \\
&[A_1(\alpha),A'^+(\beta)] = -[A'(\alpha),A_1^+(\beta)] = \frac{\eta}{\xi}(\alpha,\beta) \quad .
\end{aligned} \qquad 52$$

<u>The Hamiltonian</u>

I want to show that the Hamiltonian operator $\mathcal{H}$ of eq. 20 can be written in terms of our A and A^+ as

$$\mathcal{H} = m\sum_i A_1^+(e_i)A_1(e_i) + \frac{1}{2} : V\left\{\frac{2}{m}\sum_i A'^+(e_i)A'(e_i)\right\} : \quad , \qquad 53$$

where : : indicates normal ordering, i.e. absorption operators stand to the right of creation operators. I have to prove that the matrix elements of the operator 53 coincide with the r.h.s. of eq. 20.

$$m\sum_{i=1}^{\infty}\left(A_1(e_i)\Phi[f;g'],A_1(e_i)\Phi[f,g]\right)$$

$$= m\sum_i (h;e_i)(e_i,h)\left(\Phi[f;g'],\Phi[f,g]\right) = \frac{1}{2}(f'-img;f-img)\left(\Phi[f;g'],\Phi[f,g]\right.$$

It remains to evaluate the matrix elements of the second term.

$$A'(e_i)\Phi[f,g] = A'(e_i)N\sum_{p,q}\frac{1}{\sqrt{p!q!}}\prod_{i=1}^{p}h(\underline{x}_i)\prod_{i=p+1}^{p+q}\hbar(\underline{x}_i)$$

$$= \frac{-i}{\sqrt{\xi}}\{n(e_i,h)-(e_i,\hbar)\}\Phi[f,g] = -\sqrt{\frac{m}{2}}\zeta(e_i,g)\Phi[f,g]$$

Hence

$$\sum_{i_1,\dots,i_n=1}^{\infty}\left(\{\prod_{j=1}^{n}A'(e_{i_j})\}\Phi[f;g'],\{\prod_{j=1}^{n}A'(e_{i_j})\}\Phi[f,g]\right)$$

$$= \{\frac{m}{2}\zeta^2\sum_i(g;e_i)(e_i,g)\}^n(\Phi[f;g'],\Phi[f,g])$$

and

$$(\Phi[f;g'],H\Phi[f,g])$$

$$= \frac{1}{2}[(f'-img;f-img)+V\{\zeta^2(g;g)\}](\Phi[f;g'],\Phi[f,g]) \quad ,$$

which is the desired result.

Expression for the contributions to $\Phi[f,g]$ from the irreducible subspaces

If we replace in the commutators that appear in 52 $A'(\alpha)$ by

$$A_2(\alpha) = A(0\oplus\alpha) \quad , \tag{54}$$

their values will remain unchanged except for those in the last line, which will vanish. Thus the only nonvanishing commutators are

$$[A_1(\alpha),A_1^+(\beta)] = [A_2(\alpha),A_2^+(\beta)] = (\alpha,\beta) \quad . \tag{55}$$

The contributions to $\Phi[f,g]$ from the irreducible subspaces which are listed in table 40 in the column "original notation" can be constructed as follows by letting A_1^+ and A_2^+ act on the vacuum Φ_0.

$\mathcal{H}_0$		N
$\mathcal{H}_1$	1	$NA_1^+(h)\Phi_0$
	2	$NA_2^+(\hbar)\Phi_0$
$\mathcal{H}_2$	11	$\frac{N}{2!}(A_1^+(h))^2\Phi_0$
	12	$\frac{N}{2!}\{A_1^+(h)A_2^+(\hbar)+A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\frac{N}{2!}\{A_1^+(h)A_2^+(\hbar)-A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	22	$\frac{N}{2!}(A_2^+(\hbar))^2\Phi_0$
$\mathcal{H}_3$	111	$\frac{N}{3!}(A_1^+(h))^3\Phi_0$
	112	$\frac{N}{3!}\{(A_1^+(h))^2A_2^+(\hbar)+2A_1^+(h)A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	$\begin{smallmatrix}1&1\\2&\end{smallmatrix}$	$\frac{N}{3!}\{2(A_1^+(h))^2A_2^+(\hbar)-2A_1^+(h)A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	122	$\frac{N}{3!}\{A_1^+(h)(A_2^+(\hbar))^2+2A_1^+(\hbar)A_2^+(\hbar)A_2^+(h)\}\Phi_0$
	$\begin{smallmatrix}1&2\\2&\end{smallmatrix}$	$\frac{N}{3!}\{2A_1^+(h)(A_2^+(\hbar))^2-2A_1^+(\hbar)A_2^+(\hbar)A_2^+(h)\}\Phi_0$
	222	$\frac{N}{3!}(A_2^+(\hbar))^3\Phi_0$
$\mathcal{H}_4$	1111	$\frac{N}{4!}(A_1^+(h))^4\Phi_0$
	1112	$\frac{N}{4!}\{(A_1^+(h))^3A_2^+(\hbar)+3(A_1^+(h))^2A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	$\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}$	$\frac{N}{4!}\{3(A_1^+(h))^3A_2^+(\hbar)-3(A_1^+(h))^2A_1^+(\hbar)A_2^+(h)\}\Phi_0$
	1122	$\frac{N}{4!}\{(A_1^+(h))^2(A_2^+(\hbar))^2+4A_1^+(h)A_1^+(\hbar)A_2^+(\hbar)A_2^+(h)+(A_1^+(\hbar))^2(A_2^+(h))^2\}\Phi_0$
	$\begin{smallmatrix}1&1&2\\2&&\end{smallmatrix}$	$\frac{N}{4!}\{3(A_1^+(h))^2(A_2^+(\hbar))^2-3(A_1^+(\hbar))^2(A_2^+(h))^2\}\Phi_0$
	$\begin{smallmatrix}1&1\\2&2\end{smallmatrix}$	$\frac{N}{4!}\{2(A_1^+(h))^2(A_2^+(\hbar))^2-4A_1^+(h)A_1^+(\hbar)A_2^+(\hbar)A_2^+(h)+2(A_1^+(\hbar))^2(A_2^+(h))^2\}\Phi_0$
	1222	$\frac{N}{4!}\{A_1^+(h)(A_2^+(\hbar))^3+3A_1^+(\hbar)(A_2^+(\hbar))^2A_2^+(h)\}\Phi_0$
	$\begin{smallmatrix}1&2&2\\2&&\end{smallmatrix}$	$\frac{N}{4!}\{3A_1^+(h)(A_2^+(\hbar))^3-3A_1^+(\hbar)(A_2^+(\hbar))^2A_2^+(h)\}\Phi_0$
	2222	$\frac{N}{4!}(A_2^+(\hbar))^4\Phi_0$

Note the beautiful regularities of the expressions, which can make us guess what the expression would be in any other case.

$\Phi_n[f,g]$ and $\Phi[f,g]$ can be constructed as follows:

$$\Phi_n[f,g] = N\frac{\{A_1^+(h)+A_2^+(\hbar)\}^n}{n!}\Phi_o \tag{57}$$

and

$$\Phi[f,g] = Ne^{A_1^+(h)+A_2^+(\hbar)}\Phi_o \quad . \tag{58}$$

Expressions in terms of the A's for $\phi(f)$, $\pi(g)$, and $U[f,g]$

$\phi(e)$ and $\pi(e)$ have the matrix elements

$$(\Phi[f';g'],\phi(e)\Phi[f,g])$$
$$= \frac{i}{\sqrt{2m}}\{(e,h)+\eta(e,\hbar)-(h;e)-\eta(\hbar;e)\}(\Phi[f';g'],\Phi[f,g])$$

and

$$(\Phi[f';g'],\pi(e)\Phi[f,g]) = \sqrt{\frac{m}{2}}\{(e,h)+(h;e)\}(\Phi[f';g'],\Phi[f,g]) \quad .$$

Since $A_1(e)\Phi[f,g] = (e,h)\Phi[f,g]$ and $A_2(e)\Phi[f,g] = (e,\hbar)\Phi[f,g]$, it follows that

$$\phi(e) = \frac{i}{\sqrt{2m}}\{A_1(e)+\eta A_2(e)-A_1^+(e)-\eta A_2^+(e)\} \quad , \tag{59}$$

$$\pi(e) = \sqrt{\frac{m}{2}}\{A_1(e)+A_1^+(e)\} \quad . \tag{60}$$

Hence

$$U[f,g] = e^{i\{\phi(f)-\pi(g)\}} = e^{A_1^+(h)+A_2^+(\hbar)-A_1(h)-A_2(\hbar)}$$
$$= Ne^{A_1^+(h)+A_2^+(\hbar)}e^{-A_1(h)-A_2(\hbar)} \quad , \tag{61}$$

where the Baker-Hausdorff relation * has been used in the last step. Using $\Phi[f,g]=U[f,g]\Phi_o$, the above eq. lets

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* The Baker-Hausdorff relation states that two operators A and B whose commutator $C=[A,B]$ satisfies $[C,A]=[C,B]=0$ fulfil $e^Ae^B=e^{A+B}e^{\frac{1}{2}[A,B]}$. This is discussed e.g. in reference 9.

us recover eq. 58.

The beauty of the formulation of RS-models in terms of creation and annihilation operators and the close connection between them and the operators used in references 1 to 3 suggests that it might be worthwile to try to reformulate these theories using A's and A^+'s from the very beginning.

9 Conclusion

I would like to summarize what I could add in this chapter to the results that Klauder presented at the Brandeis summer school. I recognized that weak convergence in $\mathfrak{h}$, $\psi_n \to \psi$, implies in $\mathcal{H}$ $\frac{\Psi_n}{N_{\Psi_n}} \to \frac{\Psi}{N_\Psi}$, not $\Psi \underset{n}{\to} \Psi$. This was essential for being able to exploit disjointness of $\mathcal{H}$ more fully: We were able to exclude dependence of

$$\frac{(\Phi[f;g'],\mathcal{B}\Phi[f,g])}{(\Phi[f;g'],\Phi[f,g])}$$

on (f,f),(f,g),(g,g) and on(f;f'),(f;g'),(g;g') in a very simple manner. We recognized then that the form, eq. 38, for $(\Phi[f;g'],\mathcal{B}\Phi[f,g])$ which ensues implies that the invariant subspaces of $\mathcal{H}$ found in the previous chapter were in fact irreducible. We know now the subspace structure of $\mathcal{H}$ considered as a representation space for $O(\infty,\mathbb{R}_3)$. It does not quite agree with what the authors of reference 2 expected in their footnote 13. In that paper, $\mathcal{H}$ was calculated explicitely in the first and second sectors for an $\mathcal{H}$ corresponding to a specific classical Hamiltonian. The ground work for an investigation also of the higher sectors has now been laid. Finally the formulation of the theory in terms of creation and absorption operators has been expanded, and in particular expressions for the contributions to $\Phi[f,g]$ from the various irreducible subspaces have been given.

VI CELL MODELS OF THE FIRST KIND *

1 Introduction

We saw that every interaction in the case of RS-models was badly non-local. By this I mean that the relation between values of the field at arbitrarily distant points plays a role in $H(f,g)$ whenever it contains terms involving the field to a higher power than the second. We will now consider a class of models corresponding to less restricted classical Hamiltonians, which will include interacting models where the relation between the values of the field at points more than, say, 10^{-16}cm apart will not matter.

We divide $\mathbb{R}_3$ into a countable infinity of cells numbered 1,2,... We can do this e.g. in such a way that points more than 10^{-16}cm apart necessarily lie in different cells. It will sometimes be convenient to make a more special choice. Consider a Cartesian coordinate system in $\mathbb{R}_3$. It may be oblique and different length units may be chosen on different axes. We take as cells the unit parallelepipeds whose centers have integer coordinates. These coordinates can be used to refer to the cells, i.e. each $\underline{i} = (i_1, i_2, i_3) \in \mathbb{Z}_3$ corresponds to a

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* Prof. J.R.Klauder suggested to me to investigate this problem. I enjoyed numerous discussions with him on its solution.

cell. We can go back to the original way of numbering the cells by assigning 1 to the cell around the origin, the numbers 2 to 3^3 in some definite way to the surrounding cells whose centers have no coordinate with absolute value >1, 3^3+1 to 5^3 to the cells in the next layer, etc.

For $h \in L^2(\mathbb{R}_3)$ and $i = 1,2,...$ we define

$$h_i(\underline{x}) = \begin{cases} h(\underline{x}) & \text{if } \underline{x} \text{ lies in cell } i \\ 0 & \text{otherwise .} \end{cases}$$

The $h_i(\underline{x})$ form the Hilbert space $L^2(i)$ of the L^2-functions that vanish outside cell i. We want to investigate the quantum field theories that correspond to classical Hamiltonians of the form

$$H(f,g) = \frac{1}{2}[(f,f)+\{\sum_i m_{i_o}^2 (g_i,g_i)\}+V_o\{z_i = (g_i,g_i)\}] \quad , \qquad 63'$$

where $f,g \in L_r^2(\mathbb{R}_3)$ and where V_o can be expanded into a power series in its countably many arguments z_i. We can assume without loss of generality that V_o contains no linear terms in its arguments because we can collect them all in $\sum_i m_{i_o}^2 (g_i,g_i)$. We shall be most interested in the cases where $m_{i_o} = m_o \; \forall i$. We call the models corresponding to a classical Hamiltonian of the form 63' cell models of the 1^{st} kind or shortly C_1-models.

Let us determine the linear operators T that map every $f \in L_r^2(\mathbb{R}_3)$ into $Tf \in L_r^2(\mathbb{R}_3)$ and that satisfy $H(f,g)=H(Tf,Tg)$ $\forall$ $H(f,g)$ of the form 63'. A necessary and sufficient restriction on the T is that they obey $\forall\ f\epsilon L_r^2(\mathbb{R}_3)$, $i=1,2,..$

$$Tf_i \epsilon L_r^2(i) \quad , \quad (Tf_i,Tf_i) = (f_i,f_i) \quad .$$

These T form a group $\mathcal{T}'$ which is the direct product

$$\mathcal{T}' = \prod_{i=1}^{\infty} \otimes \mathcal{T}_i \qquad 64'$$

of the groups $\mathcal{T}_i$ whose elements T_i are defined by

$$T_i f(\underline{x}) = \begin{cases} Tf(\underline{x}) & \text{if } \underline{x}\epsilon \text{ cell } i \\ f(\underline{x}) & \text{otherwise} \end{cases}$$

$\forall\ f(\underline{x})\epsilon L_r^2(\mathbb{R}_3)$ and an appropriate $T\epsilon\mathcal{T}'$. The structure of each group $\mathcal{T}_i$ is very close to $O(\infty,\mathbb{R}_3)$; an appropriate alternative symbol for $\mathcal{T}_i$ would be $O(\infty,i)$. However none of the $\mathcal{T}_i$ and not even $\mathcal{T}'$ contain the Euclidean group as a subgroup. Since Euclidean invariance is desirable on physical grounds, we will often restrict ourselves to models that possess at least lattice translation invariance:

$$H(f,g) = \frac{1}{2}[(f,f)+m_o^2(g,g)+V_o\{z_{\underline{i}}=(g_{\underline{i}},g_{\underline{i}})\}]$$

where V_o satisfies 63"

$$V_o\{z_{\underline{i}}=(g_{\underline{i}},g_{\underline{i}})\} = V_o\{z_{\underline{i}}=(g_{\underline{i}+\underline{n}},g_{\underline{i}+\underline{n}})\} \quad \forall\ \underline{n}\epsilon\mathbb{Z}_3 \quad .$$

In this case we use the division into unit parallelepipeds. We have now the bigger invariance group

$$\mathcal{T} = \mathcal{T}' \otimes \mathcal{T}'' \quad , \qquad 64''$$

where $\mathcal{T}''$ is the group whose elements $T(\underline{n}),\underline{n}\epsilon\mathbb{Z}_3$ are defined by

$$f(\underline{x}) = \big(T(\underline{n})f\big)(\underline{x}+\underline{n}) \quad .$$

We call $\mathcal{T}''$ the group of lattice translations.

2 Quantum Theory Assumptions

The assumptions (i)',(ii)', and (iii)' that we make for the models C_I' without lattice translation invariance can be gained from those for RS-models by replacing everywhere $O(\infty,\mathbb{R}_3)$ by $\mathcal{T}'$.

Although not much has changed in the assumptions, the derivation of results similar to the theorems 1 and 2 for RS-models, which we quoted in chapter III, is now much more cumbersome. We would expect the difficulty of deriving such results if $O(\infty,\mathbb{R}_3)$ is everywhere replaced by $\mathcal{T}$ to be somewhere in between since $\mathcal{T}' \subset \mathcal{T} \subset O(\infty,\mathbb{R}_3)$. However, it turns out that the lattice translation invariant case would be by far the most involved for the mentionned choice of assumptions. To simplify life we choose instead:

The assumptions (i)",(ii)", and (iii)" that we make for the models C_I'' with lattice translation invariance can be gained from those for RS-models by replacing $O(\infty,\mathbb{R}_3)$ by $\mathcal{T}$ with the exception that we assume: Up to a constant there is only one vector which is invariant under all the $U[T]$ with $T \in \mathcal{T}'$ (not $T \in \mathcal{T}$).

In the following I will just quote the results that I was able to derive from these assumptions since they can be proved using methods similar to those that I will describe in the next chapter.

3 The Reproducing Kernel

Theorem 1' *

Every reproducing kernel satisfying (i)' and (ii)' is of the form

$$(\Phi_o, \Phi[f,g]) = e^{-\frac{1}{4}\sum_i \{\frac{\xi_i}{m_i}(f_i,f_i)+m_i(g_i,g_i)\}} \qquad 65'$$

where $m_i>0$, $\xi_i \geq 1$ $\forall$ i. If $\exists$ $\bar{m}>\underline{m}>0$ and $\bar{\xi}>1$ such that $\underline{m} \leq m_i \leq \bar{m}$ and $\xi_i \leq \bar{\xi}$, then we are assured of the existence of representations of the CCR satisfying (i)' and (ii)' and having a reproducing kernel of the form 65'. Representations corresponding to the same set $\{m_i, \xi_i | i=1,2,...\}$ are unitarily equivalent and representations corresponding to different sets are inequivalent. The representations are irreducible if all $\xi_i=1$ and reducible otherwise.

Theorem 1"

Every reproducing kernel satisfying (i)" and (ii)" is of the form

$$(\Phi_o, \Phi[f,g]) = e^{-\frac{1}{4}\{\frac{\xi}{m}(f,f)+m(g,g)\}} \quad \text{with } m>0,\ \xi \geq 1 \quad . \qquad 65''$$

The pairs (m,ξ) satisfying the above restrictions are in one to one correspondence with the equivalence classes of CCR representations that satisfy (i)" and (ii)". The representations are irreducible if $\xi=1$, reducible if $\xi>1$.

* Verboven considers in reference 10 CCR representations of this type for the description of an infinite system of harmonic oscillators in thermal equilibrium at the temperature T. Only the restricted class of kernels of type 65' where the ξ_i are determined by the m_i and T is relevant for his problem.

We recognize that for C_1''-models the same representations of the CCR appear as for RS-models.

4 The Irreducible Subspaces of $\mathcal{H}$

Using eq. 65' one finds

$$(\Phi[f';g'],\Phi[f,g]) = N'N e^{\sum_{i=1}^{\infty}\{(h_i',h_i)+(\hbar_i',\hbar_i)\}} ,$$

where

$$h_i = \frac{f_i - i m_i g_i}{\sqrt{2m_i}} \quad \text{and} \quad \hbar_i = \frac{\sqrt{\xi_i - 1}\; f_i}{\sqrt{2m_i}} .$$

It follows that $\mathcal{H}$ is a direct sum of subspaces $\mathcal{H}_n$,

$$\mathcal{H} = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n .$$

Consider $\mathcal{H}$ as a representation space for $\mathcal{T}'$. To each sequence $i_1, i_2, \ldots, i_n$ of n integers $\neq 0$ with

$$i_1 = i_2 = \ldots = i_{m_1} < i_{m_1+1} = \ldots = i_{m_1+m_2} < \ldots < i_{n-m_p+1} = \ldots = i_n$$

there corresponds an invariant subspace $\mathcal{H}^{i_1\ldots i_n}$ of $\mathcal{H}_n$, which contributes

$$N \frac{(h_{i_1} \oplus \hbar_{i_1}) \cdot \ldots \cdot (h_{i_n} \oplus \hbar_{i_n})}{\sqrt{m_1!} \cdot \ldots \cdot \sqrt{m_p!}}$$

towards $\Phi[f,g]$. All the $\mathcal{H}^{i_1\ldots i_n}$ are disjoint. The irreducible subspaces of $\mathcal{H}^{i_1\ldots i_n}$ are in one to one correspondence with the sequences of p standard tableaux consisting of $m_1, m_2, \ldots$, and m_p squares respectively and filled with ones and twos if the ξ corresponding to the tableau is >1 and with ones only if $\xi=1$. Two of the representations contained in $\mathcal{H}^{i_1\ldots i_n}$ are equivalent if and only if the corresponding sequences of Young graphs coincide. $\mathcal{H}^{1227}$ e.g. contains if $\xi_1=1$, $\xi_2>1$, and $\xi_7>1$ the irreducible subspaces

$$\mathcal{H}^{\boxed{1}\,\boxed{1\,1}\,\boxed{1}} \quad \mathcal{H}^{\boxed{1}\,\boxed{1\,2}\,\boxed{1}} \quad \mathcal{H}^{\boxed{1}\,\boxed{2\,2}\,\boxed{1}} \qquad \mathcal{H}^{\boxed{1}\,\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\,\boxed{1}}$$

$$\mathcal{H}^{\boxed{1}\,\boxed{1\,1}\,\boxed{2}} \quad \mathcal{H}^{\boxed{1}\,\boxed{1\,2}\,\boxed{2}} \quad \mathcal{H}^{\boxed{1}\,\boxed{2\,2}\,\boxed{2}} \qquad \mathcal{H}^{\boxed{1}\,\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\,\boxed{2}}$$

The six spaces on the left are equivalent and so are the two on the right.

5 Operators Commuting with $U[\mathcal{G}']$ or with $U[\mathcal{G}]$

If $\mathcal{B}\in\{U[\mathcal{G}']\}'$, then its matrix elements will have the form

$$(\Phi[f;g'],\mathcal{B}\Phi[f,g]) = N'N\sum_n \frac{1}{n!}\sum_{j_1,\ldots,j_n=1}^{\infty}\ \sum_{\left.\begin{matrix}l_1,\ldots,l_n\\ m_1,\ldots,m_n\end{matrix}\right\}=1,2}$$

$$b(j_1,l_1,m_1;\ldots;j_n,l_n,m_n)(h'^{\,l_1}_{j_1},h^{m_1}_{j_1})\ldots(h'^{\,l_n}_{j_n},h^{m_n}_{j_n}) \quad , \qquad 66$$

where $h^1_j=h_j$ and $h^2_j=\hbar_j$. We can require without restricting the $\mathcal{B}$ that

$$b(j_1,l_1,m_1;\ldots;j_n,l_n,m_n) = b(j_{i_1},l_{i_1},m_{i_1};\ldots;j_{i_n},l_{i_n},m_{i_n})$$

for all permutations $i_1,\ldots,i_n$ of $1,\ldots,n$. The summation over l_i and m_i is in fact restricted to $l_i=m_i=1$ if $\xi_{j_i}=1$ since scalar products involving $h^2_{j_1}=\hbar_{j_1}$ vanish then.
For $\mathcal{B}\in\{U[\mathcal{T}]\}'$ the $b(\ldots)$'s have the further property

$$b(\underline{j}_1,l_1,m_1;\ldots;\underline{j}_n,l_n,m_n) = (\underline{j}_1+\underline{n},l_1,m_1;\ldots;\underline{j}_n+\underline{n},l_n,m_n)$$
$$\forall\ \ \underline{n}\in\mathbb{Z}_3$$

If we insist that $\mathcal{B}$ be bounded, then the magnitude of the $b(\ldots)$'s will be restricted too. This last requirement is simplest for irreducible representations of the CCR, where the b(...)'s must be uniformly bounded. To $\mathcal{B}$=I corresponds

$$b(j_1,l_1,m_1;\ldots;b_n,l_n,m_n) = \prod_{i=1}^{n}\delta_{l_i m_i} \quad .$$

6 The Hamiltonian

The situation is very simple for irreducible representations of the CCR, where for a given set of m_i with $0<\underline{m}\leq m_i\leq\bar{m}$ the only $\mathcal{H}$ that satisfies (iii)' has the matrix elements

$$\left(\Phi[f';g'],\mathcal{H}\Phi[f,g]\right)=\frac{1}{2}\sum_i(f_i'-im_ig_i',f_i-im_ig_i)\left(\Phi[f';g'],\Phi[f,g]\right) \quad .$$

Similarly we have for theories obeying (i)",(ii)" and (iii)" for given m only one $\mathcal{H}$:

$$\left(\Phi[f';g'],\mathcal{H}\Phi[f,g]\right) = \frac{1}{2}(f'-img';f-img)\left(\Phi[f';g'],\Phi[f,g]\right) \quad .$$

Theorem 2'

For each $\{m_i,\xi_i\}$ representation with $0<\underline{m}\leq m_i\leq\bar{m}$ and $1\leq\xi_i\leq\bar{\xi}$ $\forall i$ $\exists$ a self-adjoint operator $\mathcal{H}$ satisfying the assumptions (iii)' with matrix elements

$$(\Phi[f';g'],\mathcal{H}\Phi[f,g])$$

$$=\frac{1}{2}[\sum_i(f_i'-im_ig_i',f_i-im_ig_i)+V\{z_i=\zeta_i^2(g_i',g_i)\}](\Phi[f';g'],\Phi[f,g]) \qquad 67'$$

for a big class of $V\{z_i\}$; $\zeta_i^2=\frac{\xi_i-1}{\xi_i}$. A sufficient restriction on the $V\{z_i\}$ to ensure the existence of $\mathcal{H}$ is

$$V\{z_i\}=\sum_{p=1}^{p_{max}}\sum_E\sum_{\substack{n_1,\ldots,n_p\\=1,2,\ldots}}v_{n_1\ldots n_p}^{i_1\ldots i_p}(z_{i_1})^{n_1}\cdot\ldots\cdot(z_{i_p})^{n_p}.$$

E denotes summation over those $i_1,\ldots,i_p$ that satisfy $i_1<\ldots<i_p$ and for which $\zeta_{i_j}>0$ for $j=1,\ldots,p$. The $v_{n_1\ldots n_p}^{i_1\ldots i_p}$ are real, of modulus smaller than some constant, such that for each $i=1,2,\ldots$ only a finite number of those $v_{n_1\ldots n_p}^{i_1\ldots i_p}$ that contain i among $i_1,\ldots,i_p$ do not vanish, and such that

$$\sum_{p=1}^{p_{max}}\sum_E\sum_{\substack{n_1,\ldots,n_p\\=1,2,\ldots}}v_{n_1\ldots n_p}^{i_1\ldots i_p}\prod_{j=1}^{p}\left(\frac{2}{m_{i_j}}\right)^{n_j}\frac{q_{i_j}!}{(q_{i_j}-n_j)!}>0$$

$\forall$ sets $(q_1,q_2,\ldots)$ of nonnegative integers q_i except $(0,0,\ldots)$.

In order to formulate the corresponding theorem for C_1''-theories we have to explain what we mean by $\underline{i}<\underline{j}$.

$\underline{i}<\underline{j}$ means $i_1<j_1$ or $i_1=j_1,i_2<j_2$ or $i_1=j_1,i_2=j_2,i_3<j_3$. 68

Note that $\underline{i}<\underline{j}$ does not imply that the corresponding i and j satisfy $i<j$ and vice versa.

Theorem 2"

For each reproducing kernel in eq. 65" ∃ a Hamiltonian $\mathcal{H}$ satisfying (iii)" and with matrix elements

$$(\Phi[f;g'],\mathcal{H}\Phi[f,g])$$

$$=\frac{1}{2}[(f'-img;f-img)+V\{z_{\underline{i}}=\zeta^2(g'_{\underline{i}},g_{\underline{i}})\}](\Phi[f;g'],\Phi[f,g]) \qquad 67"$$

for every

$$V\{z_{\underline{i}}\} = \sum_{p=1}^{p_{max}} \sum_{\underline{i}_1<\ldots<\underline{i}_p} \sum_{\substack{n_1,\ldots,n_p\\=1,2,\ldots}} v^{\underline{i}_1\ldots\underline{i}_p}_{n_1\ldots n_p}(z_{\underline{i}_1})^{n_1}\cdot\ldots\cdot(z_{\underline{i}_p})^{n_p}$$

with real $v^{\underline{i}_1\ldots\underline{i}_p}_{n_1\ldots n_p} = v^{\underline{i}_1+\underline{n}\ldots\underline{i}_p+\underline{n}}_{n_1\ \ldots\ n_p}$ $\forall$ $\underline{n}\in\mathbb{Z}_3$, of modulus smaller than some constant, such that for each $\underline{i}\in\mathbb{Z}_3$ only a finite number of those $v^{\underline{i}_1\ldots\underline{i}_p}_{n_1\ldots n_p}$ that contain $\underline{i}$ among $\underline{i}_1,\ldots,\underline{i}_p$ do not vanish, and such that

$$\sum_{p=1}^{p_{max}} \sum_{\underline{i}_1<\ldots<\underline{i}_p} \sum_{\substack{n_1,\ldots,n_p\\=1,2,\ldots}} v^{\underline{i}_1\ldots\underline{i}_p}_{n_1\ldots n_p} \prod_{j=1}^{p} \left(\frac{2}{m_{\underline{i}_j}}\right)^{n_j} \frac{q_{\underline{i}_j}!}{(q_{\underline{i}_j}-n_j)!}>0$$

$\forall$ sets $(q_1,q_2,\ldots)$ of nonnegative integers q_i except $(0,0,\ldots)$.

Although we shall not bother to derive a similar sufficient condition for the existence of Hamiltonians for our next model, we shall not give the proof here because it is lengthy and not very illuminating.

The diagonal matrix elements of the Hamiltonians in theorem 2' have the form of the classical Hamiltonian if we put

$$V\{z_i\} = V_o\{z_i\}+\sum_i v_1^i z_i \quad , \quad m_{io}^2 = m_i^2+v_1^i \zeta_i^2 \quad .$$

The situation is similar for the C_1'' case.

7 Discussion

The results for these models are very similar to those for RS-models. In particular, the Hamiltonian has also here no continuous spectrum. This is connected with the fact that the T that leave H(f,g) invariant act independently in each cell, which implies that all the $\mathcal{H}^{i_1 \ldots i_n}$ are disjoint and, therefore, that $\mathcal{H}$ contains only a finite number of equivalent irreducible representations.

To get Hamiltonians with a continuous spectrum, we will study in the next chapter quantum theories corresponding to more general classical Hamiltonians that are only invariant under transformations which are determined completely once their action in any one of the cells is known.

VIII CELL MODELS OF THE SECOND KIND

1 Introduction

In order to describe the classical theories that we want to quantize, we have again to divide space into cells. We will use the unit parallelepipeds introduced in the beginning of the last chapter. The C_2-models, that we want to consider now, correspond to classical Hamiltonians of the form

$$H(f,g) = \frac{1}{2}[(f,f)+V'\{z_{\underline{ij}} = (g_{\underline{i}},g_{\underline{j}})\}] \quad . \qquad 69a$$

V' depends on a countably infinite number of arguments, to which I refer by $z_{\underline{ij}}$, $\underline{i},\underline{j}\in\mathbb{Z}_3$.

$$(h_{\underline{i}},h_{\underline{j}}) = \int_0 h^*(\underline{i}+\underline{x})h(\underline{j}+\underline{x})d\underline{x} \qquad 70$$

$\forall\ h\in L^2(\mathbb{R}_3)$, where $\int_0$ means that we integrate only over values $\underline{x}$ that lie in cell $\underline{0}$, i.e. the one which contains the origin of the coordinate system. We restrict ourselves to $V'\{z_{\underline{ij}}\}$ satisfying

$$V'\{z_{\underline{ij}}=(g_{\underline{i}},g_{\underline{j}})\} = V'\{z_{\underline{ij}}=(g_{\underline{i}+\underline{n}},g_{\underline{j}+\underline{n}})\} \qquad 69b$$

$\forall\ g\in L^2_r(\mathbb{R}_3)$ and $\underline{n}\in\mathbb{Z}_3$. Again we require that V' can be expanded into a power series in its arguments. Note that this time I did not bother to split the terms that are linear in the scalar products off $V'\{z_{\underline{ij}}\}$. We will call $H(f,g)$ "free" if $V'\{z_{\underline{ij}}\}$ is linear in its arguments.

We denote the group of linear operators $T:L^2_r(R_3)\to L^2_r(R_3)$ that satisfy $H(f,g)=H(Tf,Tg)$ $\forall$ $H(f,g)$ of the form 69a by $\mathcal{T}'_{new}$ and later again by $\mathcal{T}'$ although this refers now to a smaller group than in the last chapter. Given any $T\epsilon\mathcal{T}_i$ $\exists$ $T'\epsilon\mathcal{T}'_{new}$ that acts in cell $\underline{i}$ exactly as T does. T determines T' uniquely in contrast to what we had previously. Assume $g'(\underline{j}+\underline{x})=g(\underline{i}+\underline{x})$ $\forall$ $\underline{i}+\underline{x}\epsilon$ cell $\underline{i}$, then $(T'g')(\underline{j}+\underline{x})=(T'g)(\underline{i}+\underline{x})$ $\forall$ $\underline{i}+\underline{x}\epsilon$ cell $\underline{i}$.

Thanks to property 69b, H(f,g) is again invariant under $\mathcal{T}''$ and therefore under

$$\mathcal{T} = \mathcal{T}' \otimes \mathcal{T}'' \quad .$$

THE REPRODUCING KERNEL

2 Provisional Assumptions about the Reproducing Kernel

We will initially not deal with arbitrary $f,g\epsilon L^2_r(\mathbb{R}_3)$ but only with such that vanish outside a finite number of cells. We refer to these shortly as $f,g\epsilon L$. We make the assumptions:

(i) $\forall$ $f\epsilon L$ $\exists$ two self-adjoint operators $\phi(f)$ and $\pi(f)$ acting on a separable Hilbert space $\mathcal{H}$ with positive definite scalar product and satisfying

$$\phi(cf)=c\phi(f) \quad , \quad \pi(cf)=c\pi(f) \quad \forall \text{ real } c \tag{71}$$

and such that

$$U[f,g] = e^{i\{\phi(f)-\pi(g)\}} \tag{72}$$

fulfils

$$U[f';g']U[f,g] = e^{\frac{i}{2}\{(f';g)-(g';f)\}} U[f'+f,g'+g] \tag{73}$$

(ii) $\exists$ a vector $\Phi_0 \epsilon \mathscr{H}$ such that finite linear combinations of vectors of the form

$$\Phi[f,g] = U[f,g]\Phi_0$$

are dense in $\mathscr{H}$.

$\forall$ $T \epsilon \mathscr{T}$ $\exists$ a unitary transformation $U[T]$ with the property

$$U[T]\Phi[f,g] = \Phi[Tf,Tg] \quad , \tag{74}$$

and $\exists$ an antiunitary transformation $\mathfrak{J}$ such that

$$\mathfrak{J}\Phi[f,g] = \Phi[-f,g] \quad . \tag{75}$$

Up to a constant there is only one vector which is invariant under all the $U[T]$ with $T \epsilon \mathscr{T}'$.

3 Functional Form and Continuity of the Reproducing Kernel

I shall in the more detailled calculations that follow sometimes indicate relations that have been used by giving their number underneath the equality sign. We shall often find it convenient to label the cells 1,2,..., as described in the introduction to chapter VI. (h_i, h_j) tells you to replace i and j by the corresponding $\underline{i}$ and $\underline{j}$ and to apply definition 70.

First we will prove

Lemma 1

$\Phi[f,g]$ is strongly continuous in the product topology

$$d(\tilde{f},\tilde{g};f,g) = \|\tilde{f}-f\| + \|\tilde{g}-g\| = \sqrt{(\tilde{f}-f,\tilde{f}-f)} + \sqrt{(\tilde{g}-g,\tilde{g}-g)} \quad . \tag{76}$$

Let $f,g\in L$; $\varepsilon>0$ be given and introduce the abbrevations

$$W[f] = U[f,0] \quad \text{and} \quad V[g] = U[0,g] \quad .$$

$$\|\Phi[\tilde f,\tilde g]-\Phi[f,g]\| = \|e^{\frac{i}{2}(\tilde f,\tilde g)} V[\tilde g]W[\tilde f]\Phi_0 - e^{\frac{i}{2}(f,g)} V[g]W[f]\Phi_0\|$$

$$\leq \|e^{\frac{i}{2}(\tilde f,\tilde g)} V[\tilde g]W[\tilde f]\Phi_0 - e^{\frac{i}{2}(f,g)} V[\tilde g]W[\tilde f]\Phi_0\|$$

$$+\|e^{\frac{i}{2}(f,g)} V[\tilde g]W[\tilde f]\Phi_0 - e^{\frac{i}{2}(f,g)} V[g]W[\tilde f]\Phi_0\|$$

$$+\|e^{\frac{i}{2}(f,g)} V[g]W[\tilde f]\Phi_0 - e^{\frac{i}{2}(f,g)} V[g]W[f]\Phi_0\|$$

$$= |e^{\frac{i}{2}(\tilde f,\tilde g)} - e^{\frac{i}{2}(f,g)}| + \|(V[\tilde g]-V[g])\Phi[\tilde f,0]\| + \|(W[\tilde f]-W[f])\Phi_0\|$$

We can write uniquely

$$f = af^1 \quad , \quad \tilde f = (a+\alpha)f^1+\beta f^2 \quad ,$$

$$g = bg^1 \quad , \quad \tilde g = (b+\gamma)g^1+\delta g^2 \quad ,$$

where $(f^i,f^j) = (g^i,g^j) = \delta_{ij} \quad , \quad i,j = 1,2 \quad .$

$$T_1 = |e^{\frac{i}{2}(\tilde f,\tilde g)} - e^{\frac{i}{2}(f,g)}|$$

$$= |e^{\frac{i}{2}\{(af^1,\gamma g^1+\delta g^2)+(\alpha f^1+\beta f^2,bg^1)+(\alpha f^1+\beta f^2,\gamma g^1+\delta g^2)\}} -1|$$

$T_1 \approx |\frac{i}{2}\{\ \}|$ for small $\alpha,\beta,\gamma,\delta$. Thus $\exists$ $\delta_1>0$ such that $T_1<\frac{\varepsilon}{3}$ whenever $\sqrt{\alpha^2+\beta^2} + \sqrt{\gamma^2+\delta^2}<\delta_1$.

Using the fact that $V[g']V[g] = V[g'+g]$, we find

$$T_2 = \|(V[\tilde g]-V[g])\Phi[\tilde f,0]\|$$

$$= \|(V[\gamma g^1+\delta g^2]-1)V[bg^1]\Phi[\tilde f,0]\| \leq \|V[\gamma g^1+\delta g^2]-1\| \quad .$$

Because $V[cg]=e^{-ic\pi(g)}$ implies that $V[cg]$ is weakly continuous in the real parameter c, there exists $\delta_2>0$ such that $T_2<\frac{\varepsilon}{3}$ whenever $\sqrt{\gamma^2+\delta^2} < \delta_2$.

Similarly one shows that $\exists\ \delta_3>0$ such that
$T_3=\|(W[\tilde{f}]-W[f])\Phi_0\|<\frac{\varepsilon}{3}$ whenever $\sqrt{\alpha^2+\beta^2}<\delta_3$.
Hence $\|\Phi[\tilde{f},\tilde{g}]-\Phi[f,g]\|<\varepsilon$ whenever
$\sqrt{(\tilde{f}-f,\tilde{f}-f)}+\sqrt{(\tilde{g}-g,\tilde{g}-g)} = \sqrt{\alpha^2+\beta^2}+\sqrt{\gamma^2+\delta^2}<\delta'$,
where $\delta'=\min(\delta_1,\delta_2,\delta_3)$.

Let us now investigate the reproducing kernel $(\Phi[f',g'],\Phi[f,g])$. Since

$$(\Phi[f',g'],\Phi[f,g]) = (\Phi_0,U[-f',-g']U[f,g]\Phi_0)$$

$$\underset{73}{=} e^{-\frac{i}{2}\{(f',g)-(g',f)\}}(\Phi_0,U[f-f',g-g']\Phi_0) \quad , \tag{77}$$

it suffices to consider $(\Phi_0,\Phi[f,g])$, to which we shall refer as the reproducing kernel, too.

$$(\Phi_0,\Phi[f,g]) = (\Phi_0,\Phi[Tf,Tg]) \qquad \forall\ \ T\varepsilon\mathcal{T} \tag{78}$$

because $U[T]\Phi_0=\Phi_0$. It follows, even if we restrict our attention to $T\varepsilon\mathcal{T}'$, that $(\Phi_0,\Phi[f,g])$ depends on $(f_i,f_j),(f_i,g_j),(g_i,g_j)$, $i,j=1,2,\ldots$ only. We write

$$(\Phi_0,\Phi[f,g])=\mathcal{K}\{\alpha_{ij}=(f_i,f_j),\beta_{ij}=(f_i,g_j),\gamma_{ij}=(g_i,g_j)\}$$

or shortly

$$=\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\} \quad .$$

It follows from lemma 1 that the reproducing kernel is continuous in the metric 76. We will now prove that it is also continuous in each of the arguments that appear in $\mathcal{K}$. Consider f,g and f',g' such that $(f_i,f_j)=(f'_i,f'_j)$, $(g_i,g_j)=(g'_i,g'_j)$ for $i\leqslant j$ and $(f_i,g_j)=(f'_i,g'_j)\ \forall\ i,j$

with the exception that one of the primed scalar products is ε^2 bigger than the corresponding unprimed one. I will prove that $\exists$ $\tilde{f},\tilde{g}$ and $\tilde{f}',\tilde{g}'$ with the properties that both pairs lead to the same set of scalar products and that $d(\tilde{f},\tilde{g};f,g)+d(\tilde{f}',\tilde{g}';f',g')\leqslant 4\varepsilon$. It follows then from the lemma that $(\Phi_0,\Phi[f',g'])$ comes as close to $(\Phi_0,\Phi[f,g])$ as we like if we choose $\varepsilon>0$ small enough.

If $(f_k',f_k')=(f_k,f_k)+\varepsilon^2$ for a certain k, then we can take $\tilde{f}=f+\varepsilon h_k,\tilde{g}=g$ and $\tilde{f}'=f',\tilde{g}'=g'$, where $h_k\in L_r^2(k)$, $(h_k,h_k)=1$ and $(h_k,f_i)=(h_k,g_i)=0$ for $i=1,2,\ldots$. The situation is similar if $(g_k',g_k')=(g_k,g_k)+\varepsilon^2$ but a little more complicated in the other cases. Take e.g. $(f_k',g_l')=(f_k,g_l)+\varepsilon^2$, consider h_k,h_l,h_k',h_l' such that

$$(h_k,h_k)=(h_l,h_l)\qquad =(h_k,h_l)=1\qquad (h_i,f_j)=(h_i,g_j)=0$$

$$(h_k',h_k')=(h_l',h_l')=1\quad ,\quad (h_k',h_l')=0\qquad (h_i',f_j')=(h_i',g_j')=0$$

for $i=k,l$ and $j=1,2,\ldots$.
$\tilde{f}=f+\varepsilon h_k$, $\tilde{g}=g+\varepsilon h_l$ and $\tilde{f}'=f'+\varepsilon h_k'$, $\tilde{g}'=g'+\varepsilon h_l'$ have the required properties. The argument is similar for the cases $(f_k',f_l')=(f_k,f_l)+\varepsilon^2$ and $(g_k',g_l')=(g_k,g_l)+\varepsilon^2$ for $k<l$.

The reproducing kernel has the further properties

$$(\Phi_0,U[f,g]\Phi_0)^*=(U[f,g]\Phi_0,\Phi_0)=(\Phi_0,U^{-1}[f,g]\Phi_0)=(\Phi_0,U[-f,-g]\Phi_0)$$

and

$$(\Phi_0,U[f,g]\Phi_0)^*=(\mathcal{J}U[f,g]\Phi_0,\mathcal{J}\Phi_0)^*=(U[-f,g]\Phi_0,\Phi_0)^*=(\Phi_0,U[-f,g]\Phi_0)\quad .$$

We collect our results in

Lemma 2

Every reproducing kernel satisfying the assumptions (i) and (ii) has the functional form

$$(\phi_0,\Phi[f,g])=\mathcal{K}\{\alpha_{ij}=(f_i,f_j),\beta_{ij}=(f_i,g_j),\gamma_{ij}=(g_i,g_j)\} \quad . \qquad 79$$

$\mathcal{K}\{\ \}$ is a continuous function in each of its arguments and satisfies

$$\mathcal{K}^*\{\alpha_{ij},\beta_{ij},\gamma_{ij}\}=\mathcal{K}\{\alpha_{ij},\beta_{ij},\gamma_{ij}\}=\mathcal{K}\{\alpha_{ij},-\beta_{ij},\gamma_{ij}\} \quad . \qquad 80$$

Note that this lemma still holds if we allow for arbitrary $f,g\epsilon L^2_r(\mathbb{R}_3)$ and also if we relax the assumptions (ii) by replacing $\mathcal{T}$ there by $\mathcal{T}'$. In fact, so far I have not made use of the assumption $f,g\epsilon L$ nor of eq. 74 for $T\notin\mathcal{T}'$.

4 A More Explicit Form for the Reproducing Kernel

We will prove in this section that the reproducing kernel is necessarily of the form

$$(\phi_0,\Phi[f,g])=e^{-\sum\limits_{\underline{m}}\{A_{\underline{m}}\sum\limits_{\underline{n}}(f_{\underline{m}},f_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum\limits_{\underline{n}}(g_{\underline{m}},g_{\underline{m}+\underline{n}})\}} \quad ,$$

where $A_{\underline{m}}=A_{-\underline{m}}$ and $C_{\underline{m}}=C_{-\underline{m}}$ are real numbers.

Consider two sequences f^k and g^k, $k=1,2,\ldots$ satisfying

(1) $(f^k,g)\to 0$ and $(g^k,f)\to 0$ as $k\to\infty$ $\forall$ $f,g\epsilon L$;

(2) $U[f^k,g^k]$ converges weakly to a bounded operator A, i.e. $(\Lambda,U[f^k,g^k]\Psi)\to(\Lambda,A\Psi)$ as $k\to\infty$ $\forall$ $\Lambda,\Psi\epsilon\mathcal{H}$.

It follows from these two assumptions and

$$U[f^k,g^k]U[f,g] = e^{i[(f^k,g)-(g^k,f)]}U[f,g]U[f^k,g^k]$$

that

$$AU[f,g] = U[f,g]A \quad . \tag{81}$$

We call A the tag operator; $\|A\| \leq 1$ since $|(\Lambda, U[f^k,g^k]\Psi)| \leq \|\Lambda\|\|\Psi\|$. The silver rule, proved in appendix A, tells us that we have to verify (2) only for a total set of Λ and Ψ, e.g. only for the $(\Phi[f';g'], U[f^k,g^k]\Phi[f,g])$. Because of (1) and eq. 73 we only have to look at

$$(\Phi_0, U[f+f^k, g+g^k]\Phi_0)$$

$$= \mathcal{K}\{(f_i+f_i^k, f_j+f_j^k), (f_i+f_i^k, g_j+g_j^k), (g_i+g_i^k, g_j+g_j^k)\} \quad . \tag{82}$$

To define appropriate sequences choose $f';g' \in L$. Let $i_1<i_2<\ldots<i_m$ and $i_{m+1}<\ldots<i_{m+n}$ be the numbers of the cells in which f' and g' respectively do not vanish identically, and let $\underline{i}_r$ be the coordinate of the center of cell i_r. We can write

$$f'_{i_r}(\underline{x}) = \sum_{j=1}^{r} p_j^r \, u^j \, (\underline{x}-\underline{i}_r)$$

and

$$g'_{i_s}(\underline{x}) = \sum_{j=1}^{s} p_j^s \, u^j \, (\underline{x}-\underline{i}_s) \quad ,$$

where $u^j(\underline{x}) \in L_r^2(\underline{o})$ and $(u^i,u^j)=\delta_{ij}$ for $i,j=1,2,\ldots,m+n$. Complete this set of vectors to an orthonormal basis in $L_r^2(\underline{o})$. We choose as our sequences

$$f^k_{i_r}(\underline{x}) = \sum_{j=1}^{r} p_j^r \, u^{j+k} \, (\underline{x}-\underline{i}_r)$$

and 83

$$g^k_{i_s}(\underline{x}) = \sum_{j=1}^{s} p_j^s \, u^{j+k} \, (\underline{x}-\underline{i}_s) \quad .$$

It is obvious that they satisfy (1), but they also satisfy (2) since the finite number of arguments in 82 that do not vanish for all k all converge,

$$(\Phi_o, U[f+f^k, g+g^k]\Phi_o)$$
$$\to \mathcal{K}\{(f_i,f_j)+(f_i',f_j'),(f_i,g_j)+(f_i',g_j'),(g_i,g_j)+(g_i',g_j')\}$$
$$= (\Phi_o, U[f,g]A\Phi_o) \quad . \tag{84}$$

Because $f\to -f$, $g\to -g$ and $f\to Tf, g\to Tg$ with $T\in\mathcal{T}'$ do not change the arguments in $\mathcal{K}$, we get

$$(\Phi_o, U[f,g]A\Phi_o) = (\Phi[f,g], A\Phi_o) = (\Phi[Tf,Tg], A\Phi_o) = (\Phi[f,g], U^{-1}[T]A\Phi_o) \quad .$$

From the assumed cyclicity of the representation and the assumption that only multiples of Φ_o are invariant under all the $U[T]$ with $T\in\mathcal{T}'$ we conclude that

$$A\Phi_o = a\Phi_o \quad , \tag{85}$$

where a is a number, called the tag, which can easily be determined by putting f=g=0 in eq. 84.

$$a = \mathcal{K}\{(f_i',f_j'),(f_i',g_j'),(g_i',g_j')\}$$

Using that eq. again we find

$$\mathcal{K}\{(f_i,f_j)+(f_i',f_j'),(f_i,g_j)+(f_i',g_j'),(g_i,g_j)+(g_i',g_j')\}$$
$$= \mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}\mathcal{K}\{(f_i',f_j'),(f_i',g_j'),(g_i',g_j')\} \quad . \tag{86}$$

Notice that we cannot straight away conclude from this

$$\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}$$
$$= \prod_{i',j'} \mathcal{K}\{\alpha_{ij}=\delta_{ii'}\delta_{jj'}(f_{i'},f_{j'}) \quad ,\beta_{ij}\equiv 0 \quad ,\gamma_{ij}\equiv 0\}$$
$$\cdot\mathcal{K}\{\alpha_{ij}\equiv 0 \quad ,\beta_{ij}=\delta_{ii'}\delta_{jj'}(f_{i'},g_{j'}) \;,\gamma_{ij}\equiv 0\}$$
$$\cdot\mathcal{K}\{\alpha_{ij}\equiv 0 \quad ,\beta_{ij}\equiv 0 \quad ,\gamma_{ij}=\delta_{ii'}\delta_{jj'}(g_{i'},g_{j'})\} \quad .$$

In fact, if we take one of the factors $\mathcal{K}$ on the r.h.s., we will often find that no $f',g'\epsilon L$ exist that lead to the required values of the scalar products.

From eq. 86 follows for every positive integer n that

$$\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}=\left[\mathcal{K}\{(\tfrac{f_i}{\sqrt{n}},\tfrac{f_j}{\sqrt{n}}),(\tfrac{f_i}{\sqrt{n}},\tfrac{g_j}{\sqrt{n}}),(\tfrac{g_i}{\sqrt{n}},\tfrac{g_j}{\sqrt{n}})\}\right]^n$$

$$=\left[\mathcal{K}\{\tfrac{1}{n}(f_i,f_j),\tfrac{1}{n}(f_i,g_j),\tfrac{1}{n}(g_i,g_j)\}\right]^n \quad . \qquad 87$$

Since $\mathcal{K}$ is real, a continuous function of its arguments, and equals 1 when they all vanish, it follows first that

$$\mathcal{K}\{(\tfrac{f_i}{\sqrt{n}},\tfrac{f_j}{\sqrt{n}}),(\tfrac{f_i}{\sqrt{n}},\tfrac{g_j}{\sqrt{n}}),(\tfrac{g_i}{\sqrt{n}},\tfrac{g_j}{\sqrt{n}})\}>0$$

for big enough n and then from eq. 87 that

$$\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}>0 \qquad \forall \quad f,g\epsilon L \quad . \qquad 88$$

For any postive, rational $\frac{m}{n}$ we get

$$\mathcal{K}\{\tfrac{m}{n}(f_i,f_j),\tfrac{m}{n}(f_i,g_j),\tfrac{m}{n}(g_i,g_j)\}=\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}^{\frac{m}{n}} \quad ,$$

and invoking again the continuity of $\mathcal{K}$ it follows that

$\forall \quad t>0$

$$\mathcal{K}\{t(f_i,f_j),t(f_i,g_j),t(g_i,g_j)\}=\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}^t . \qquad 89$$

Let us investigate some special $\mathcal{K}$'s, from which we will be able to build up all the others.

$$\left.\begin{array}{l}
X_{mm}=\mathcal{K}\{\alpha_{ij}=\delta_{im}\delta_{jm} \;,\; \beta_{ij}=0 \;,\; \gamma_{ij}=0 \;\} \\
Z_{mm}=\mathcal{K}\{\alpha_{ij}=0 \;,\; \beta_{ij}=0 \;,\; \gamma_{ij}=\delta_{im}\delta_{jm} \;\}
\end{array}\right\} m=1,2,\ldots$$

$$\left.\begin{array}{l}
X^{\pm}_{mn}=\mathcal{K}\{\alpha_{ij}=\delta_{im}\delta_{jm}\pm\delta_{im}\delta_{jn}+\delta_{in}\delta_{jn},\beta_{ij}=0,\gamma_{ij}=0\} \\
Z^{\pm}_{mn}=\mathcal{K}\{\alpha_{ij}=0,\beta_{ij}=0,\gamma_{ij}=\delta_{im}\delta_{jm}\pm\delta_{im}\delta_{jn}+\delta_{in}\delta_{jn}\}
\end{array}\right\} m<n$$

$$Y^{\pm}_{mn}=\mathcal{K}\{\alpha_{ij}=\delta_{im}\delta_{jm},\beta_{ij}=\pm\delta_{im}\delta_{jn},\gamma_{ij}=\delta_{in}\delta_{jn}\} \quad m,n=1,2,\ldots$$

Note that f's and g's leading to the above values of the arguments do in fact exist.

$$Y^{+}_{mn} \underset{80}{=} Y^{-}_{mn} = Y_{mn} \quad ,$$

$$Y^2_{mn} = Y^{+}_{mn} Y^{-}_{mn} \underset{86}{=} X^2_{mm} Z^2_{nn} \to \frac{Y_{mn}}{X_{mm} Z_{nn}} = 1$$

$$X^{+}_{mn} X^{-}_{mn} = X^2_{mm} X^2_{nn} \to \frac{X^{+}_{mn}}{X_{mm} X_{nn}} = \left[\frac{X^{-}_{mn}}{X_{mm} X_{nn}}\right]^{-1}$$

$$Z^{+}_{mn} Z^{-}_{mn} = Z^2_{mm} Z^2_{nn} \to \frac{Z^{+}_{mn}}{Z_{mm} Z_{nn}} = \left[\frac{Z^{-}_{mn}}{Z_{mm} Z_{nn}}\right]^{-1}$$

$$\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\} \cdot \prod_m X_{mm}^{\sum_{n\neq m}|(f_m,f_n)| + \sum_n |(f_m,g_n)|}$$

$$\cdot \prod_n Z_{nn}^{\sum_{m\neq n}|(g_m,g_n)| + \sum_m |(f_m,g_n)|}$$

$$= \prod_m X_{mm}^{(f_m,f_m)} \cdot \prod_m Z_{mm}^{(g_m,g_m)} \cdot \prod_{m,n} Y_{mn}^{|(f_m,g_n)|}$$

$$\cdot \prod_{m<n} X^{\pm}_{mn}{}^{|(f_m,f_n)|} \cdot \prod_{m<n} Z^{\pm}_{mn}{}^{|(g_m,g_n)|} \quad ,$$

where $\pm$ applies in $X^{\pm}_{mn}$ if $(f_m,f_n) \gtrless 0$, similarly for $Z^{\pm}_{mn}$. Therefore

$$\mathcal{K}\{(f_i,f_j),(f_i,g_j),(g_i,g_j)\}$$

$$= \prod_m X_{mm}^{(f_m,f_m)} \cdot \prod_m Z_{mm}^{(g_m,g_m)} \cdot \left\{\prod_{m,n} \left(\frac{Y_{mn}}{X_{mm} Z_{nn}}\right)^{|(f_m,g_n)|}\right\}$$

$$\cdot \left\{\prod_{m<n} \left(\frac{X^{\pm}_{mn}}{X_{mm} X_{nn}}\right)^{|(f_m,f_n)|}\right\} \cdot \left\{\prod_{m<n} \left(\frac{Z^{\pm}_{mn}}{Z_{mm} Z_{nn}}\right)^{|(g_m,g_n)|}\right\} \quad .$$

Since $\frac{Y_{mn}}{X_{mm} Z_{nn}} = 1$, it follows that $\mathcal{K}$ does not depend on the (f_i,g_j), and we will drop the label β_{ij}. We introduce the notation

$$X_{mm}=e^{-A_{mm}}\;,\quad Z_{mm}=e^{-C_{mm}}\;,\quad \frac{X^{+}_{mn}}{X_{mm}X_{nn}}=e^{-2A_{mn}}\;,\quad \frac{Z^{+}_{mn}}{Z_{mm}Z_{nn}}=e^{-2C_{mn}}\quad .$$

One easily convinces oneself that $A_{mm},C_{mm}>0$ and A_{mn},C_{mn} real. Putting $A_{mn}=A_{nm}$ and $C_{mn}=C_{nm}$ we can write

$$\mathcal{K}\{(f_i,f_j),(g_i,g_j)\} = e^{-\sum\limits_{m,n}\{A_{mn}(f_m,f_n)+C_{mn}(g_m,g_n)\}} \quad . \qquad 90$$

So far we have exploited our assumptions for $T\epsilon\mathfrak{T}'$ only. Now we want to take also the lattice translations into account. It will therefore be convenient to label the cells by the coordinates of their centers. Since $(\Phi_o,\Phi[f,g]) \underset{78}{=} (\Phi_o,\Phi[T(\underline{p})f,T(\underline{p})g]) \quad \forall \quad \underline{p}\epsilon\mathbb{Z}_3$, it follows that

$$\mathcal{K}\{(f_{\underline{i}},f_{\underline{j}}),(g_{\underline{i}},g_{\underline{j}})\}=e^{-\sum\limits_{\underline{m}}\{A_{\underline{m}}\sum\limits_{\underline{n}}(f_{\underline{n}},f_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum\limits_{\underline{n}}(g_{\underline{n}},g_{\underline{m}+\underline{n}})\}} \qquad 91$$

with $A_{\underline{m}},C_{\underline{m}}$ real; $A_{\underline{m}}=A_{-\underline{m}}$, $C_{\underline{m}}=C_{-\underline{m}}$; and $A_{\underline{o}}>0$, $C_{\underline{o}}>0$.

5 Sufficient Restrictions on the Reproducing Kernel

Further restrictions on the numbers $A_{\underline{m}}$ and $C_{\underline{m}}$ can be derived most conveniently if we consider $\underline{m}$ as the label of Fouriercoefficients and pass over to a continuous label. We define

$$f(\underline{x},\underline{k}) = \sum_{\underline{l}} f_{\underline{l}}(\underline{x}+\underline{l})e^{-i\underline{k}\underline{l}} \quad . \qquad 92$$

Remember that there are no convergence problems since the sum consists of a finite number of terms only. We will use the abbreviation

$$\int_{D} dk \qquad \text{for} \qquad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d\underline{k}}{(2\pi)^3} \quad .$$

D refers to the domain specified in the explicit expression.

$$\int_D dk\ f^*(\underline{x},\underline{k})g(\underline{x},\underline{k})e^{i\underline{k}\underline{n}}=\sum_{\underline{l},\underline{m}} f_{\underline{l}}(\underline{x}+\underline{l})g_{\underline{m}}(\underline{x}+\underline{m})\int_D dk\ e^{i(\underline{l}-\underline{m}+\underline{n})\underline{k}}$$

$$=\sum_{\underline{l},\underline{m}} f_{\underline{l}}(\underline{x}+\underline{l})g_{\underline{m}}(\underline{x}+\underline{m})\delta_{\underline{m},\underline{l}+\underline{n}}\quad ,$$

where $\delta_{\underline{m},\underline{l}+\underline{n}}$ is the Kronecker delta. Therefore

$$\sum_{\underline{l}} f_{\underline{l}}(\underline{x}+\underline{l})g_{\underline{l}+\underline{n}}(\underline{x}+\underline{l}+\underline{n}) = \int_D dk\ f^*(\underline{x},\underline{k})g(\underline{x},\underline{k})e^{i\underline{k}\underline{n}}\quad . \qquad 93$$

We restrict our attention to the case

$$\sum_{\underline{n}} A_{\underline{n}}^2 \ \text{ finite}\quad , \quad \sum_{\underline{n}} C_{\underline{n}}^2 \ \text{ finite}\quad .$$

Each such sequence $A_{\underline{n}}$ defines for almost every $\underline{k}$ a function $A(\underline{k})$ with the property

$$A_{\underline{n}} = \int_D dk\ A(\underline{k})e^{i\underline{k}\underline{n}}\quad .$$

We call the space of the functions $A(\underline{k})$ $L^2(D)$. Introduce $\sum_{\underline{l}}(f_{\underline{l}},f_{\underline{l}+\underline{n}}) = F_{\underline{n}}$. $\sum_{\underline{n}} F_{\underline{n}}^2$ is finite since all the summations involved contain a finite number of terms only. We can therefore apply Parseval's theorem

$$\sum_{\underline{n}} A_{\underline{n}} \sum_{\underline{l}} (f_{\underline{l}},f_{\underline{l}+\underline{n}}) = \sum_{\underline{n}} A_{\underline{n}} F_{\underline{n}} = \int_D dk\ A^*(\underline{k})F(\underline{k})\quad ,$$

where

$$F(\underline{k})=\sum_{\underline{n}} F_{\underline{n}} e^{-i\underline{k}\underline{n}} \underset{93}{=} \sum_{\underline{n}} \int_c d\underline{x} \int_D dk' |f(\underline{x},\underline{k}')|^2 e^{i(\underline{k}'-\underline{k})\underline{n}} = \int_c d\underline{x}|f(\underline{x},\underline{k})|^2\quad .$$

Thus

$$\sum_{\underline{n}} A_{\underline{n}} \sum_{\underline{l}} (f_{\underline{l}},f_{\underline{l}+\underline{n}}) = \int_D dk\ A(\underline{k})\int_c d\underline{x}|f(\underline{x},\underline{k})|^2\quad ,$$

where we used that $A(\underline{k})$ is real because $A_{\underline{n}}=A_{-\underline{n}}$. The reproducing kernel, eq. 91, can therefore be written as

$$(\Phi_0,\Phi[f,g]) = e^{-\int_D dk\int_0 d\underline{x}\{A(\underline{k})|f(\underline{x},\underline{k})|^2+C(\underline{k})|g(\underline{x},\underline{k})|^2\}} \quad . \qquad 94$$

The order in which the two integrations are carried out does not matter because $(\Phi_0,\Phi[f,g])<1$ unless $\|f\|\|g\|=0$ implies $A(\underline{k})>0$ and $C(\underline{k})>0$ almost everywhere. We put

$$C(\underline{k}) = \frac{m(\underline{k})}{4} \quad \text{and} \quad A(\underline{k}) = \frac{\xi(\underline{k})}{4m(\underline{k})} \quad .$$

Obviously we have $m(\underline{k})>0$ and $\xi(\underline{k})>0$.

$$(\Phi[f',g'],\Phi[f,g]) \underset{77}{=} e^{-\frac{i}{2}\{(f',g)-(g',f)\}}$$

$$\cdot e^{-\frac{1}{4}\int_D dk\int_0 d\underline{x}\{\frac{\xi(\underline{k})}{m(\underline{k})}|f(\underline{x},\underline{k})-f'(\underline{x},\underline{k})|^2+m(\underline{k})|g(\underline{x},\underline{k})-g'(\underline{x},\underline{k})|^2\}}$$

Using eq. 93, the abbreviation $N=(\Phi_0,\Phi[f,g])$, and the fact that $f(\underline{x},-\underline{k})=f^*(\underline{x},\underline{k})$, we find

$$(\Phi[f',g'],\Phi[f,g])$$
$$=N'Ne^{\frac{1}{2}\int_D dk\int_0 d\underline{x}\{\frac{\xi(\underline{k})}{m(\underline{k})}f'^*(\underline{x},\underline{k})f(\underline{x},\underline{k})+m(\underline{k})g'^*(\underline{x},\underline{k})g(\underline{x},\underline{k})-i[*]\}}$$

with $[*] = f'^*(\underline{x},\underline{k})g(\underline{x},\underline{k})-g'^*(\underline{x},\underline{k})f(\underline{x},\underline{k})$.

We introduce

$$h(\underline{x},\underline{k}) = \frac{f(\underline{x},\underline{k})-im(\underline{k})g(\underline{x},\underline{k})}{\sqrt{2m(\underline{k})}} \quad , \quad \hbar(\underline{x},\underline{k}) = \frac{\eta(\underline{k})f(\underline{x},\underline{k})}{\sqrt{2m(\underline{k})}} \quad , \qquad 95$$

where $\eta(\underline{k}) = \sqrt{|\xi(\underline{k})-1|}$, and we will use the abbreviation

$$(h'(\underline{k}'),h(\underline{k})) = \int_0 d\underline{x}\; h'^*(\underline{x},\underline{k}')h(\underline{x},\underline{k}) \quad . \qquad 96$$

Thus

$$(\Phi[f',g'],\Phi[f,g]) = N'Ne^{\int_D dk\{(h'(\underline{k}),h(\underline{k}))\pm(\hbar'(\underline{k}),\hbar(\underline{k}))\}} \quad , \qquad 97$$

where $\pm$ applies for $\xi(\underline{k})\gtrless 1$.

Consider first the cases where $A(\underline{k})$ and $C(\underline{k})$ are piecewise constant. By this we mean that there exists a division of D into a finite number of cells such that $A(\underline{k})$ is constant inside each cell, and that there exists a similar division for the $C(\underline{k})$. Consider the minimally finer division such that both $A(\underline{k})$ and $C(\underline{k})$ and therefore $\xi(\underline{k})$ and $m(\underline{k})$ are constant in each cell. There is a finite number N of them. We call the values of $\xi(\underline{k})$ and of $m(\underline{k})$ in cell i ($i=1,\ldots,N$) ξ_i and m_i. We can write

$$(\Phi[f',g'],\Phi[f,g]) = \prod_{i=1}^{N}(\Phi_i[f',g'],\Phi_i[f,g]) \qquad 98$$

with

$$(\Phi_i[f',g'],\Phi_i[f,g]) = N_i'N_i e^{\int_i dk\{(h'(\underline{k}),h(\underline{k}))\pm(\hbar'(\underline{k}),\hbar(\underline{k}))\}}$$

and

$$N_i=(\Phi_{oi},\Phi_i[f,g])=e^{-\frac{1}{4}\int_i dk\int_o d\underline{x}\{\frac{\xi_i}{m_i}|f(\underline{x},\underline{k})|^2+m_i|g(\underline{x},\underline{k})|^2\}} \quad , \qquad 99$$

where $\int_i dk$ indicates integration over $\underline{k}$ in cell i. Eq. 98 tells us that we may consider $\Phi[f,g]$ as an element of a Hilbert space $\bar{\mathcal{H}}$, which is the direct product of the N Hilbert spaces $\bar{\mathcal{H}}_i$, $i=1,\ldots,N$. $U[f,g]$ must therefore be of the form

$$U[f,g] = \prod_{i=1}^{N}\otimes U_i[f,g] \quad ,$$

where $U_i[f,g]$ and $U_j[f',g']$ commute if $i\neq j$. Note that assumption 73 is not in disagreement with this property since it can be written as

$$U[f';g']U[f,g] = e^{\frac{i}{2}\sum_{i=1}^{N}\int dk \int_{o} d\underline{x}\{f'^*(\underline{x},\underline{k})g(\underline{x},\underline{k}) - g'^*(\underline{x},\underline{k})f(\underline{x},\underline{k})\}} \cdot U[f'+f,g'+g] \quad . \qquad 100$$

The assumptions (i) reduce for the $U_i[f,g]$ to $(\bar{i})$:

$\forall$ $f \epsilon L$ $\exists$ two self-adjoint operators $\phi_i(f)$ and $\pi_i(f)$ acting on a separable Hilbert space $\mathcal{H}_i$ with positive definite scalar product and satisfying

$$\phi_i(cf) = c\phi_i(f) \quad , \quad \pi_i(cf) = c\pi_i(f)$$

$\forall$ real c and such that

$$U_i[f,g] = e^{i\{\phi_i(f) - \pi_i(g)\}}$$

fulfils

$$U_i[f';g']U_i[f,g] = e^{\frac{i}{2}\int dk \int_{o} d\underline{x}\{f'^*(\underline{x},\underline{k})g(\underline{x},\underline{k}) - g'^*(\underline{x},\underline{k})f(\underline{x},\underline{k})\}} \cdot U_i[f'+f,g'+g] \quad .$$

Using the same methods as Klauder in reference 1, one shows that for

$$(\Phi_{oi}, \Phi_i[f,g]) = e^{-\frac{1}{4}\int dk \int_{o} d\underline{x}\{\frac{\xi_i}{m_i}|f(\underline{x},\underline{k})|^2 + m_i|g(\underline{x},\underline{k})|^2\}}$$

to correspond to $U_i[f,g]$ satisfying $(\bar{i})$ it is necessary and sufficient that $\xi_i \geq 1$. The representation is irreducible if $\xi_i = 1$ and reducible if $\xi_i > 1$. We conclude: necessary and sufficient for $(\Phi_o, \Phi[f,g])$ in eq. 94 with piecewise constant $A(\underline{k})$ and $C(\underline{k})$ to correspond to $U[f,g]$ satisfying (i) is that $A(\underline{k})C(\underline{k}) \geq \frac{1}{16}$ $\forall$ $\underline{k}$; the representation is irreducible if $A(\underline{k})C(\underline{k}) = \frac{1}{16}$ $\forall$ $\underline{k}$ and reducible otherwise.

I make now the conjecture that this statement is true for arbitrary $A(\underline{k})$ and $C(\underline{k})$ in $L^2(D)$ if we understand $\geq$ and $=$ in the sense of almost everywhere (a.e.). This conjecture seems very reasonable in view of the facts that the expression 94 for $(\Phi_o, \Phi[f,g])$ is well defined $\forall$ such $A(\underline{k})$ and $C(\underline{k})$ and that $A(\underline{k})$ and $C(\underline{k})$ are limits a.e. of piecewise constant ones. A partial proof of this conjecture is given in appendix B. In most of what follows we will use this conjecture for continuous $A(\underline{k})$ and $C(\underline{k})$ only, where it is difficult to imagine that it could be wrong. Note also that no such conjecture needs to be made for C_1-models. The results presented in the previous chapter do therefore not depend on its validity.

Why $\frac{1}{16}$ is the "critical value" for the product $A(\underline{k})C(\underline{k})$ is illustrated by eq. 97. It tells us that the Hilbert space $\bar{\mathcal{H}}$, of which the $\Phi[f,g]$'s are elements, is the direct sum of subspaces $\mathcal{H}_n$

$$\bar{\mathcal{H}} = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n \quad . \qquad 101$$

$\mathcal{H}_n$ is the symmetrized direct product of n factors h.

$$h = \int_D dk \otimes h(\underline{k}) \qquad 102$$

and $h(\underline{k})$ is isomorphic to the direct sum of two $L^2(\underline{o})$ spaces if $16A(\underline{k})C(\underline{k}) = \xi(\underline{k}) > 1$ and to $L^2(\underline{o})$ itself if $\xi(\underline{k}) = 1$. If $\xi(\underline{k}) < 1$ it is isomorphic to what we might call the direct difference $L^2(\underline{o}) \ominus L^2(\underline{o})$, whose elements are also pairs of $L^2(\underline{o})$ elements but whose scalar product is the difference: scalar product in the first space minus scalar product in the second one.*

It follows that $\bar{\mathcal{H}}$ is a separable Hilbert space with positive definite metric if $\xi(\underline{k}) \geq 1$ for almost every $\underline{k}$ and that it has an indefinite metric otherwise. Considering the contributions that the various subspaces make to $\Phi[f,g]$ ($\mathcal{H}_1$ e.g. contributes $N(h \oplus \hbar)$, one recognizes that $\bar{\mathcal{H}}$ is spanned by the $\Phi[f,g]$, i.e. every vector of $\bar{\mathcal{H}}$ can be reached as a weak limit of finite linear combinations of $\Phi[f,g]$'s. It follows that $\bar{\mathcal{H}}$ is a realization of the Hilbert space $\mathcal{H}$ mentionned in the assumptions (i) if $\xi(\underline{k}) \geq 1$ a.e. In this case we will omit the bar over $\bar{\mathcal{H}}$ in future.

It remains to prove that the assumptions (ii) are fulfilled, too. We discussed the Hilbert space spanned by the $\Phi[f,g]$ already. I will now show that eq. 74, i.e.

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* If we had chosen

$$\hbar(\underline{x},\underline{k}) = \frac{f(\underline{x},\underline{k})}{\sqrt{2m}}$$

instead of the second eq. in 95, then the scalar product in $\not{h}(\underline{k})$ would have been the one in the first $L^2(\underline{o})$ plus $(\xi(\underline{k})-1)$ times the one in the second. Again one finds that $h(\underline{k})$ possesses a positive definite metric if $\xi(\underline{k})>1$ and an indefinite one if $\xi(\underline{k})<1$. The elements of $\not{h}(\underline{k})$ are now also for $\xi(\underline{k})=1$ of the form $h(\underline{x},\underline{k}) \oplus \hbar(\underline{x},\underline{k})$, and the scalar product is positive semidefinite. The transition to the view adopted above is made by throwing out the second $L^2(\underline{o})$ space, which does not contribute to the scalar product.

$U[T]\Phi[f,g]=\Phi[Tf,Tg]$, defines in fact for each $T\epsilon\mathcal{T}$ a unitary operator $U[T]$. Using

$$(\Phi[f';g'],\Phi[f,g]) = e^{\frac{i}{2}\{(f';g)-(g';f)\}} \cdot e^{-\sum_{\underline{m}}\{A_{\underline{m}}\sum_{\underline{n}}(f_{\underline{n}}-f'_{\underline{n}},f_{\underline{m}+\underline{n}}-f'_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum_{\underline{n}}(g_{\underline{n}}-g'_{\underline{n}},g_{\underline{m}+\underline{n}}-g'_{\underline{m}+\underline{n}})\}} , \qquad 103$$

we can easily derive from eq. 74 that

$$(\Phi[f';g'],U[T]\Phi[f,g]) =(U[T^{-1}]\Phi[f';g'],\Phi[f,g]) \quad . \qquad 104$$

This proves that $U[T]$ is linear. Eq. 74 determines therefore how $U[T]$ acts on a dense set in $\mathcal{H}$, i.e. $U[T]$ is a linear operator. 104 tells us also that $U^+[T]=U[T^{-1}]$.

$$\Phi[f,g] \underset{74}{=} U[T^{-1}]U[T]\Phi[f,g] = U[T]U[T^{-1}]\Phi[f,g]$$

Hence

$$U^+[T]U[T] = U[T]U^+[T] = 1 \quad .$$

This proves that $U[T]$ is in fact a unitary operator.

One shows in very much the same way that eq. 75, i.e. $\mathcal{J}\Phi[f,g]=\Phi[-f,g]$, defines an antiunitary operator $\mathcal{J}$. Instead of eq. 104 one gets

$$(\Phi[f';g'],\mathcal{J}\Phi[f,g]) = (\mathcal{J}\Phi[f';g'],\Phi[f,g])^* \quad ,$$

which proves that $\mathcal{J}$ is antilinear. Noticing that it follows from eq. 75 that $\mathcal{J}=\mathcal{J}^{-1}$, one readily sees that $\mathcal{J}$ is antiunitary.

Finally we have to show that only multiples of Φ_0 are invariant under the $U[T]$ $\forall$ $T\epsilon\mathcal{T}'$. Expand an arbitrary $f\epsilon L$ in terms of the orthonormal functions $u^j(\underline{x})$ that we used in eq. 83:

$$f(\underline{x}) = \sum_{\underline{i}}\sum_{j} p_j^{\underline{i}} u^j(\underline{x}-\underline{i}) \quad .$$

Define $T_k \in \mathcal{T}'$ by

$$(T_k f)(\underline{x}) = \sum_{\underline{i}}\{\sum_{j=1}^{k} p_{k+1-j}^{\underline{i}} u^j(\underline{x}-\underline{i}) + \sum_{j=k+1}^{\infty} p_j^{\underline{i}} u^j(\underline{x}-\underline{i})\} \quad .$$

$$(\Phi[f;g'],U[T_k]\Phi[f,g]) = e^{-\frac{i}{2}\{(f;T_k g)-(g;T_k f)\}}$$

$$\cdot e^{-\sum_{\underline{m}}\{A_{\underline{m}}\sum_{\underline{n}}(T_k f_{\underline{n}}-f'_{\underline{n}},T_k f_{\underline{m}+\underline{n}}-f'_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum_{\underline{n}}(T_k g_{\underline{n}}-g'_{\underline{n}},T_k g_{\underline{m}+\underline{n}}-g'_{\underline{m}+\underline{n}})\}}$$

$$= (\Phi[f;g'],\Phi_o)(\Phi_o,\Phi[f,g])$$

In the last step we used: $T_k \to 0$ for $k\to\infty$. The silver rule tells us that $U[T_k]$ converges weakly towards the projection operator on the space spanned by Φ_o. Let Λ be an arbitrary vector in $\mathcal{H}$ and Ψ one that is invariant under all the $U[T_k]$.

$$(\Lambda,\Psi)=(\Lambda,U[T_k]\Psi)=\lim_{k\to\infty}(\Lambda,U[T_k]\Psi)=(\Lambda,\Phi_o)(\Phi_o,\Psi) \;\to\; \Psi=(\Phi_o,\Psi)\Phi_o \quad .$$

This proves that only multiples of Φ_o are invariant under all the $U[T]$, $T\in\mathcal{T}'$.

6 What have we Gained so far?

Theorem 1a

> Every CCR representation satisfying the assumptions (i) and (ii) has a reproducing kernel of the form
>
> $$(\Phi_o,\Phi[f,g])=e^{-\sum\limits_{\underline{m}}\{A_{\underline{m}}\sum\limits_{\underline{n}}(f_{\underline{n}},f_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum\limits_{\underline{n}}(g_{\underline{n}},g_{\underline{m}+\underline{n}})\}}$$
>
> with $A_{\underline{m}}, C_{\underline{m}}$ real, $A_{\underline{m}}=A_{-\underline{m}}$, $C_{\underline{m}}=C_{-\underline{m}}$, and $A_{\underline{o}}>0$, $C_{\underline{o}}>0$. If $\sum\limits_{\underline{m}}A_{\underline{m}}^2$ finite and $\sum\limits_{\underline{m}}C_{\underline{m}}^2$ finite, the reproducing kernel can be written as
>
> $$(\Phi_o,\Phi[f,g])=e^{-\frac{1}{4}\int_D d\underline{k}\int_o d\underline{x}\{\frac{\xi(\underline{k})}{m(\underline{k})}|f(\underline{x},\underline{k})|^2+m(\underline{k})|g(\underline{x},\underline{k})|^2\}} \qquad 105$$
>
> with $m(\underline{k})>0$, $\xi(\underline{k})\geq 1$ a.e.; $m(\underline{k})\in L^2(D)$, $\frac{\xi(\underline{k})}{m(\underline{k})}\in L^2(D)$.
> Every functional satisfying eq. 105 is the reproducing kernel of a CCR representation that satisfies (i) and (ii). It is irreducible if $\xi(\underline{k})=1$ a.e. and reducible otherwise. Two representations are equivalent if and only if the pairs $\{m(\underline{k}),\xi(\underline{k})\}$ that characterize the corresponding reproducing kernels are equal a.e.

We have not yet proved the last statement. The proof in one direction has been given by Naimark on p. 242 of reference 11, where he showed that two representations with the same reproducing kernel are equivalent. It remains to show that representations with different reproducing kernels are necessarily inequivalent. We do this with the help of the tag operator introduced earlier.

$$AU[f,g]\Phi_0 \underset{81}{=} U[f,g]A\Phi_0 \underset{85}{=} aU[f,g]\Phi_0$$

The tags a that we considered depend on the $f', g' \epsilon L$ that we used to define the tag operators A:

$$a = e^{-\frac{1}{4}\int_D dk \int_0 d\underline{x} \{\frac{\xi(\underline{k})}{m(\underline{k})} |f'(\underline{x},\underline{k})|^2 + m(\underline{k})|g'(\underline{x},\underline{k})|^2\}} .$$

One finds in the same manner as Klauder on p. 1675 of reference 1 that tags corresponding to the same choice of f' and g' are equal for two equivalent representations. This completes the proof since one can always find a pair $f', g' \epsilon L$ for which the tags are different unless the $\{m(\underline{k}), \xi(\underline{k})\}$ are for both representations equal a.e.

7 Can we Allow for More General f and g?

The aim of this section is to show that in the cases where $\sum_{\underline{k}} |A_{\underline{k}}|$ and $\sum_{\underline{k}} |C_{\underline{k}}|$ are finite we can extend the definition of U[f,g] to all $F, G \epsilon L^2_r(\mathbb{R}_3)$, and that the properties assumed in (i) and (ii) remain valid for this larger class of U[F,G]. We will refer to the assumptions (i) and (ii) in which f,g,f',g' have been replaced by F,G,F',G' by (i)' and (ii)'.

Weak convergence of the sequences $U[f^N, g^N]$

Consider for $F \epsilon L^2_r(\mathbb{R}_3)$ the sequence f^N where

$$f^N(\underline{x}) = \begin{cases} F(\underline{x}) & \text{if} \quad |\underline{x}| \leq N \\ 0 & \text{if} \quad |\underline{x}| > N \end{cases} .$$

f^N has the important properties $f^N \epsilon L$ $\forall$ N and $f^N \to F$ for

$N\to\infty$. Using the eqs. 73 and 103 we find

$$(\Phi[f';g'],U[f^N,g^N]\Phi[f,g])=e^{\frac{i}{2}[(f^N,g)-(g^N,f)-(f';g^N+g)+(g';f^N+f)]}$$

$$\cdot e^{-\sum_{\underline{m}}A_{\underline{m}}\sum_{\underline{n}}(f^N_{\underline{n}}+f_{\underline{n}}-f'_{\underline{n}},f^N_{\underline{m}+\underline{n}}+f_{\underline{m}+\underline{n}}-f'_{\underline{m}+\underline{n}})}$$

$$\cdot e^{-\sum_{\underline{m}}C_{\underline{m}}\sum_{\underline{n}}(g^N_{\underline{n}}+g_{\underline{n}}-g'_{\underline{n}},g^N_{\underline{m}+\underline{n}}+g_{\underline{m}+\underline{n}}-g'_{\underline{m}+\underline{n}})} \quad .$$

The r.h.s. converges for $N\to\infty$. This is least obvious for the term $\Sigma_{\underline{m}}A_{\underline{m}}\,\Sigma_{\underline{n}}(f^N_{\underline{n}},f^N_{\underline{m}+\underline{n}})$ and for a similar one involving the $C_{\underline{m}}$. But also these converge absolutely since

$$\sum_{\underline{m},\underline{n}}|A_{\underline{m}}(F_{\underline{n}},F_{\underline{m}+\underline{n}})|=\sum_{\underline{m}}|A_{\underline{m}}|\sum_{\underline{n}}|(F_{\underline{n}},F_{\underline{m}+\underline{n}})|\leq(\sum_{\underline{m}}|A_{\underline{m}}|)(\sum_{\underline{n}}(F_{\underline{n}},F_{\underline{n}})) \quad ,$$

where the last expression, which we found using the Schwartz inequality, is finite. Since the $U[f^N,g^N]$ are uniformly bounded and since the $\Phi[f,g]$ form a total set, the silver rule states that the unitary operators $U[f^N,g^N]$ converge weakly towards a linear operator, called $U[F,G]$, with $\|U[F,G]\|\leq 1$, and whose matrix elements between states of our total set are

$$(\Phi[f';g'],U[F,G]\Phi[f,g])=e^{\frac{i}{2}[(F,g)-(G,f)-(f';G+g)+(g';F+f)]}$$

$$\cdot e^{-\sum_{\underline{m}}A_{\underline{m}}\sum_{\underline{n}}(F_{\underline{n}}+f_{\underline{n}}-f'_{\underline{n}},F_{\underline{m}+\underline{n}}+f_{\underline{m}+\underline{n}}-f'_{\underline{m}+\underline{n}})} \qquad 106$$

$$\cdot e^{-\sum_{\underline{m}}C_{\underline{m}}\sum_{\underline{n}}(G_{\underline{n}}+g_{\underline{n}}-g'_{\underline{n}},G_{\underline{m}+\underline{n}}+g_{\underline{m}+\underline{n}}-g'_{\underline{m}+\underline{n}})} \quad .$$

Are the assumptions (i)' and (ii)' satisfied?

In order to prove

$$U[F';G']U[F,G] = e^{\frac{i}{2}\{(F';G)-(G';F)\}} U[F'+F,G'+G] \quad , \qquad 107$$

it is sufficient to show that the matrix elements of both sides between states of a total set coincide.

$$(\Phi[f';g'],U[F';G']U[F,G]\Phi[f,g])=\lim_{N\to\infty}(\Phi[f';g'],U[F';G']U[f^N,g^N]\Phi[f,g])$$

$$=\lim_{N\to\infty} e^{\frac{i}{2}\{(f^N,g)-(g^N,f)\}}(\Phi[f';g'],U[F';G']\Phi[f^N+f,g^N+g])$$

$$=e^{\frac{i}{2}\{(F,g)-(G,f)+(F';G+g)-(G';F+f)-(f';G'+G+g)+(g';F'+F+f)\}}$$

$$\cdot e^{-\sum_{\underline{m}} A_{\underline{m}} \sum_{\underline{n}} (F'_{\underline{n}}+F_{\underline{n}}+f_{\underline{n}}-f'_{\underline{n}},F'_{\underline{m}+\underline{n}}+F_{\underline{m}+\underline{n}}+f_{\underline{m}+\underline{n}}-f'_{\underline{m}+\underline{n}})}$$

$$\cdot e^{-\sum_{\underline{m}} C_{\underline{m}} \sum_{\underline{n}} (G'_{\underline{n}}+G_{\underline{n}}+g_{\underline{n}}-g'_{\underline{n}},G'_{\underline{m}+\underline{n}}+G_{\underline{m}+\underline{n}}+g_{\underline{m}+\underline{n}}-g'_{\underline{m}+\underline{n}})} \quad .$$

Using eq. 106 we find the same expression also for the matrix elements of the r.h.s. of eq. 107.

Next we want to show that $U[F,G]$ is unitary. We remark that $U[f^N,g^N]\to U[F,G]$ implies $U^+[f^N,g^N]\to U^+[F,G]$. However $U^+[f^N,g^N]=U[-f^N,-g^N]\to U[-F,-G]$. Eq. 107 tells us that $U[F,G]U[-F,-G]=U[-F,-G]U[F,G]=1$. Hence

$$U[F,G]U^+[F,G] = U^+[F,G]U[F,G] = 1 \quad ,$$

which proves the desired result.

Consider the matrix elements of $W[c_N F]=U[c_N F,0]$ for a convergent sequence of real numbers $c_N\to c$. It is easy to see that $(\Phi[f';g'],W[c_N F]\Phi[f,g])$ converges for $N\to\infty$

$\forall$ $f;g;f,g\in L$. The silver rule yields $W[c_N F]\to W[cF]$. $W[cF]$ is therefore weakly continuous in c $\forall$ $c\in\mathbb{R}$. Stone's theorem assures us that there exists a self-adjoint operator $\phi(F)$ satisfying $\phi(cF)=c\phi(F)$ and $W[F]=e^{i\phi(F)}$. Similarly we prove the existence of a self-adjoint operator $\pi(G)$ with $\pi(cG)=c\pi(G)$ and $V[G]=U[0,G]=e^{-i\pi(G)}$.

It follows from eq. 107 that

$$e^{-ic\pi(G)}e^{ic\phi(F)} = e^{-ic^2(F,G)}e^{ic\phi(F)}e^{-ic\pi(G)}$$

Comparing quadratic terms in c we find

$$[\phi(F),\pi(G)] = i(F,G) \quad .$$

The Baker-Hausdorff formula states therefore that

$$e^{-\frac{i}{2}(F,G)}\, e^{i\phi(F)}\, e^{-i\pi(G)} = e^{i\{\phi(F)-\pi(G)\}} \quad .$$

Eq. 107 tells us that the l.h.s. equals $U[F,G]$, This completes the proof that the $U[F,G]$ satisfy all the assumptions (i)'.

That also the assumptions (ii)' are satisfied can be shown in the same way as before for (ii). Therefore we want to make two remarks only.

Since we dealt also in this section exclusively with states that can be considered as weak limits of finite linear combinations of $\Phi[f,g]$'s, it follows that the Hilbert space $\mathcal{H}$ has not been enlarged. Already the $\Phi[f,g]$ span $\mathcal{H}$, the $\Phi[F,G]=U[F,G]\Phi_0$ do so a fortiori.

That eq. 103 remains true if we replace f;g;f,g by the corresponding capitals is a consequence of eq. 107 and of the unitarity of U[F,G].

Can we again change from the discrete variable $\underline{n}$ to the continuous $\underline{k}$?

First I want to show

$$\sum_{\underline{l}}(F_{\underline{l}},G_{\underline{l}+\underline{n}}) = \int_{o}d\underline{x}\int_{D}dk\ F^*(\underline{x},\underline{k})G(\underline{x},\underline{k})e^{i\underline{k}\underline{n}}$$

To do this I make use of Beppo Levi's theorem which is stated e.g. on p. 36 of reference 12.

Every series $\sum_n k_n(x)$ of summable functions for which $\sum_n \int_a^b |k_n(x)|dx$ converges, converges itself almost everywhere to a summable function and the series can be integrated term by term.

This theorem still holds if the integrations extend to infinity and also for several variables. I will give the name Beppo Levi's theorem to this generalization:

Every series $\sum_n k_n(\underline{x})$ of summable functions for which $\sum_n\int_{\mathcal{D}}|k_n(\underline{x})|d\underline{x}$ converges, converges itself a.e. to a summable function and the series can be integrated term by term, i.e. $\int_{\mathcal{D}}d\underline{x}(\sum_n k_n(\underline{x}))=\sum_n(\int_{\mathcal{D}}d\underline{x}\ k_n(\underline{x}))$.

The assumptions of the theorem are satisfied for $\sum_n\int_{\mathcal{D}}d\underline{x}\ k_n(\underline{x}) = \sum_{\underline{l}}\int_{o}d\underline{x}\ F_{\underline{l}}(\underline{x}+\underline{l})G_{\underline{l}+\underline{n}}(\underline{x}+\underline{l}+\underline{n})$ since the Schwartz inequality tells us

$$\Big(\sum_{\underline{l}}\int_0 d\underline{x}\,|F_{\underline{l}}(\underline{x}+\underline{l})G_{\underline{l}+\underline{n}}(\underline{x}+\underline{l}+\underline{n})|\Big)^2 \leqslant \sum_{\underline{l}}\int_0 |F_{\underline{l}}(\underline{x}+\underline{l})|^2 d\underline{x}\circ\sum_{\underline{l}}\int_0 |G_{\underline{l}}(\underline{x}+\underline{l})|^2 d\underline{x}$$

$$= (F,F)\cdot(G,G) \quad .$$

Therefore

$$\sum_{\underline{l}}(F_{\underline{l}},G_{\underline{l}+\underline{n}}) = \int_0 d\underline{x}\sum_{\underline{l}}F_{\underline{l}}(\underline{x}+\underline{l})G_{\underline{l}+\underline{n}}(\underline{x}+\underline{l}+\underline{n}) \quad .$$

We define $\mathcal{G}_{\underline{l}}(\underline{x})=G_{\underline{l}+\underline{n}}(\underline{x}+\underline{n})$ and use Parseval's theorem

$$\sum_{\underline{l}}F_{\underline{l}}(\underline{x})G_{\underline{l}+\underline{n}}(\underline{x}+\underline{n}) = \sum_{\underline{l}}F_{\underline{l}}(\underline{x})\mathcal{G}_{\underline{l}}(\underline{x}) = \int_D dk\; F^*(\underline{x},\underline{k})\,\mathcal{G}(\underline{x},\underline{k}) \quad ,$$

where

$$F_{\underline{l}}(\underline{x}+\underline{l}) = \int_D dk\; F(\underline{x},\underline{k})e^{i\underline{k}\underline{l}}$$

and

$$\mathcal{G}_{\underline{l}}(\underline{x}+\underline{l})=\int_D dk\,\mathcal{G}(\underline{x},\underline{k})e^{i\underline{k}\underline{l}}=G_{\underline{l}+\underline{n}}(\underline{x}+\underline{l}+\underline{n})=\int_D dk\big(G(\underline{x},\underline{k})e^{i\underline{k}\underline{n}}\big)e^{i\underline{k}\underline{l}} \quad ,$$

i.e. $\mathcal{G}(\underline{x},\underline{k})=G(\underline{x},\underline{k})e^{i\underline{k}\underline{n}}$. This completes the proof that

$$\sum_{\underline{l}}(F_{\underline{l}},G_{\underline{l}+\underline{n}}) = \int_0 d\underline{x}\int_D dk\; F^*(\underline{x},\underline{k})G(\underline{x},\underline{k})e^{i\underline{k}\underline{n}} \quad . \qquad 108$$

Thus

$$\sum_{\underline{n}}A_{\underline{n}}\sum_{\underline{l}}(F_{\underline{l}},F_{\underline{l}+\underline{n}}) = \sum_{\underline{n}}\int_0 d\underline{x}\int_D dk\,|F(\underline{x},\underline{k})|^2 A_{\underline{n}}e^{i\underline{k}\underline{n}} \quad .$$

Since we assumed $\sum_{\underline{n}}|A_{\underline{n}}|$ finite, the assumptions of Beppo Levi's theorem are again satisfied, and we may interchange the summation with the integrations. Because $\sum_{\underline{n}}A_{\underline{n}}e^{i\underline{k}\underline{n}}=A^*(\underline{k})=A(\underline{k})$ we find that

$$(\Phi_0,\Phi[F,G]) = e^{-\sum_{\underline{m}}\{A_{\underline{m}}\sum_{\underline{n}}(F_{\underline{n}},F_{\underline{m}+\underline{n}})+C_{\underline{m}}\sum_{\underline{n}}(G_{\underline{n}},G_{\underline{m}+\underline{n}})\}}$$

$$= e^{-\int_0 d\underline{x}\int_D dk\{A(\underline{k})|F(\underline{x},\underline{k})|^2+C(\underline{k})|G(\underline{x},\underline{k})|^2\}} \quad .$$

Summary *

Theorem 1b

Every functional

$$(\Phi_o,\Phi[F,G]) = e^{-\frac{1}{4}\int_{\mathcal{D}} d\underline{k}\int_o d\underline{x}\{\frac{\xi(\underline{k})}{m(\underline{k})}|F(\underline{x},\underline{k})|^2+m(\underline{k})|G(\underline{x},\underline{k})|^2\}}$$ 109

with $m(\underline{k})>0$, $\xi(\underline{k})\geq 1$ and $\sum_{\underline{n}}|\int_{\mathcal{D}} d\underline{k}\ m(\underline{k})e^{i\underline{k}\underline{n}}|$ finite, $\sum_{\underline{n}}|\int_{\mathcal{D}} d\underline{k}\frac{\xi(\underline{k})}{m(\underline{k})}e^{i\underline{k}\underline{n}}|$ finite is the reproducing kernel of a representation that satisfies (i)' and (ii)'. The statements of theorem 1a on reducibility and equivalence hold also here.

THE HAMILTONIAN

From now onwards we will consider only the representations that satisfy the assumptions (i)' and (ii)' and that possess a reproducing kernel of the type considered in theorem 1b. Since confusion is no longer possible, we will in future use small letters for the elements of $L^2(\mathbb{R}_3)$. I may

---

* We had to restrict the $A_{\underline{n}}$ and $C_{\underline{n}}$ more and more as we went along. Recently I guessed that had we restricted the f and g instead to elements of the test function space $\mathcal{S}(\mathbb{R}_3)$ we would have been able to consider for all our purposes $A(\underline{k})$ and $C(\underline{k})$ in the much larger class of tempered distributions. I do not think that the restriction to $f\in\mathcal{S}(\mathbb{R}_3)$ is objectionable on physical grounds. This means that the expected gain in elegance of the result would have been acquired at no costs. This different choice of assumptions would have allowed us to make use of some of the mathematical methods familiar to axiomatic field theorists and described e.g. in reference 13.

also occasionally quote an equation that I wrote down for $f,g\epsilon L$ if it is clear from the preceding section that the relation remains true for $f,g\epsilon L^2_r(\mathbb{R}_3)$.

8 Assumptions about the Hamiltonian

(iii)' $\exists$ a self-adjoint operator $\mathcal{H}\geq 0$ such that under suitable domain conditions

$$[U[T],\mathcal{H}] = 0 \qquad \forall\ T\epsilon\mathcal{T} \tag{110}$$

and

$$[\phi(f),\mathcal{H}] = i\pi(f) \quad . \tag{111}$$

The eq.

$$\mathcal{H}\Phi = 0 \tag{112}$$

has up to a constant a unique solution.

It is easy to see that the vector Φ of eq. 112 must be a multiple of Φ_o.

$$0 = \mathcal{H}\Phi = U[T]\mathcal{H}\Phi = \mathcal{H}U[T]\Phi \rightarrow U[T]\Phi = \alpha(T)\Phi \quad ,$$

where $\alpha(T)$ is a complex number of modulus one. It follows from the assumption expressed in eq. 74 that $U[T]U[T']=U[TT']$. The $\alpha(T)$ form therefore a one-dimensional representation of the group $\mathcal{T}'$. However, $\mathcal{T}'$ has only the trivial one-dimensional representation. Thus $U[T]\Phi=1 \quad \forall\ T\epsilon\mathcal{T}'$. According to the last of the assumptions (ii)' this implies $\Phi=c\Phi_o$.

9 Operators Commuting with $U[\mathcal{T}']$ or with $U[\mathcal{T}]$

We have to investigate the reducibility of $\mathcal{H}$ viewed as a space carrying a representation of the group $\mathfrak{T}'$. This can best be done by looking at the realization of $\mathcal{H}$ introduced in section 5. We will sometimes find it convenient to put

$$h(\underline{x},\underline{k}) = \tilde{h}(\underline{x},\underline{k}) \quad \text{and} \quad \hbar(\underline{x},\underline{k}) = \tilde{h}(\underline{x},\underline{k}+(2\pi,o,o)) \tag{113}$$

and to define a domain D' by

$$\underline{k}\epsilon D' \quad \text{if} \quad \begin{cases} \underline{k}\epsilon D \\ \text{or} \quad (\underline{k}-(2\pi,o,o))\epsilon D \quad \text{and} \quad \xi(\underline{k}-(2\pi,o,o))>1 \end{cases}$$

Eq. 97 becomes in this new notation

$$(\Phi[f;g'],\Phi[f,g]) = N'Ne^{\int_{D'} dk(\tilde{h}'(\underline{k}),\tilde{h}(\underline{k}))} \quad . \tag{114}$$

From

$$(\Phi[f;g'],U[T]\Phi[f,g]) = N'Ne^{\int_{D'} dk(\tilde{h}'(\underline{k}),(T\tilde{h})(\underline{k}))}$$

we can conclude that each of the subspaces $\mathcal{H}_n$ introduced in eq. 101 is invariant. They are further reducible except $\mathcal{H}_o$. Consider e.g.

$$\mathcal{H}_1 = \mathscr{h} = \int_{D'} dk \oplus \tilde{\mathscr{h}}(\underline{k}) \quad . \tag{115}$$

The contribution to $\Phi[f,g]$ from $\tilde{\mathscr{h}}(\underline{k})$ can be written as

$$\Phi_{\underline{k}}[f,g] = N\tilde{h}(\underline{x},\underline{k}')\delta(\underline{k}'-\underline{k}) \quad .$$

This implies that the contribution from $\mathcal{H}_1$ is

$$\Phi_1[f,g] = \int_{D'} dk'N\tilde{h}(\underline{x},\underline{k}')\delta(\underline{k}'-\underline{k}) = N\tilde{h}(\underline{x},\underline{k})$$

and that

$$(\Phi_1[f;g'],\Phi_1[f,g]) = N'N\int_{D'} dk \ (\tilde{h}'(\underline{k}),\tilde{h}(\underline{k})) \quad .$$

Notice $\Phi_1[f,g]\epsilon\mathcal{H}$ in contrast to $\Phi_{\underline{k}}[f,g]$, which cannot lie in $\mathcal{H}$ because of the δ-function it involves, which makes its norm infinite. Each element of $\tilde{h}(\underline{k})$ can be written as $h(\underline{x})\delta(\underline{k}'-\underline{k})$ with $h(\underline{x})\varepsilon L^2(\underline{o})$. It is transformed by $T\varepsilon\mathcal{T}'$ into $(Th)(\underline{x})\delta(\underline{k}'-\underline{k})$ since T acts similarly in each cell as explained in the introduction to this chapter. $\tilde{\mathcal{h}}(\underline{k})$ is therefore invariant and the representation of $\mathcal{T}'$ it contains is equivalent to the one in $L^2(\underline{o})$. We conclude that all the $\tilde{\mathcal{h}}(\underline{k})$ carry equivalent and irreducible representations of $\mathcal{T}'$.

In the higher sectors of $\mathcal{H}$ we can, too, recognize certain subspaces as being invariant. The spaces we have in mind can be denoted by $\mathcal{H}^{\underline{k}_1,\underline{k}_2,\ldots\underline{k}_n}$ with $\underline{k}_1 \leqslant \underline{k}_2 \leqslant \ldots \leqslant \underline{k}_n$, where

$$\underline{k}<\underline{k}' \quad \text{means} \quad \begin{cases} & k_1<k_1' \\ \text{or} & k_1 = k_1',\ k_2<k_2' \\ \text{or} & k_1 = k_1',\ k_2 = k_2',\ k_3<k_3' \quad . \end{cases} \qquad 116$$

They contribute $N\prod_{i=1}^{n}\tilde{h}(\underline{x}_i,\underline{k}_i')\delta(\underline{k}_i'-\underline{k}_i)$ towards $\Phi[f,g]$. The Young theory will often state that these subspaces are further reducible. We will soon return to this point.

The connection between weak convergence in $\mathcal{h}$ and in $\mathcal{H}$ is the same as for RS-models. To prove disjointness of the $\mathcal{H}_n$, we choose an orthonormal basis in $L^2(\underline{o})$. An eq. similar to 35 defines then how T_m acts in $L^2(\underline{o})$ and, since $T_m\varepsilon\mathcal{T}'$, how it acts in all of $L^2(\mathbb{R}_3)$. The rest of the proof is very similar to the one for the RS-case.

The matrix elements $(\Phi[f;g'],\mathcal{B}\Phi[f,g])$ of $\mathcal{B}\varepsilon\{U[\mathcal{T}']\}'$ can only depend on the

$$(\tilde{h}'(\underline{k}'),\tilde{h}'(\underline{k})),(\tilde{h}'(\underline{k}'),\tilde{h}(\underline{k})),(\tilde{h}(\underline{k}'),\tilde{h}(\underline{k})) \quad .$$

The methods that were employed in the RS-case lead here to

$$(\Phi[f;g'],\mathcal{B}\Phi[f,g])$$

$$= N'N\sum_n \frac{1}{n!}\{\prod_{i=1}^{n}\int\limits_D\int\limits_D dk_i' dk_i(\tilde{h}'(\underline{k}_i'),\tilde{h}(\underline{k}_i))\}b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) \quad . \qquad 117a$$

Instead of restricting the dk_i' integrations to $\underline{k}_1'\leq\underline{k}_2'\leq\ldots\leq\underline{k}_n'$, we have put the factor $\frac{1}{n!}$ and require

$$b(\underline{k}_{i_1}',\underline{k}_{i_1},\ldots,\underline{k}_{i_n}',\underline{k}_{i_n}) = b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) \qquad 117b$$

for all permutations $i_1,\ldots,i_n$ of $1,\ldots,n$. For $\mathcal{B}$ to be in fact a bounded operator, certain conditions on the magnitude of the $b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n)$ must be fulfilled. To $\mathcal{B}=I$ corresponds $b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n)=\prod_{i=1}^{n}\delta(\underline{k}_i'-\underline{k}_i)$.

$\mathcal{H}^{\underline{p}_1,\underline{p}_2,\ldots,\underline{p}_n}$ is according to Young's theory further reducible. There corresponds an invariant subspace of $\mathcal{H}^{\underline{p}_1,\underline{p}_2,\ldots,\underline{p}_n}$ to each tableau which contains $\underline{p}_1,\underline{p}_2,\ldots,\underline{p}_n$ in the standard way. We understand "standard" here in the generalized sense introduced on p. 30: The $\underline{p}_i$ do not decrease from left to right and increase downwards. To decrease and to increase are used here e.g. in the sense of eq. 116.

How many independent parameters will be needed to describe a $\mathfrak{B}_c\{U[\mathfrak{F}']\}'$ that maps $\mathcal{H}^{\underline{p}_1,\dots,\underline{p}_n}$ into itself if the invariant subspaces that we have found are in fact irreducible? We gave the answer on p. 24. The number equals the number of permutations of $\underline{p}_1,\dots,\underline{p}_n$, i.e.

$$\frac{n!}{\prod_{i=1}^{l} n_i!}$$

where l denotes the number of different $\underline{p}$ that appear among $\underline{p}_1,\dots,\underline{p}_n$ and the n_i state how often they appear. Thus $\sum_{i=1}^{l} n_i = n$. On the other hand we can show that the number of independent $b(\underline{k}_1',\underline{k}_1,\dots,\underline{k}_n',\underline{k}_n)$ in eq. 117a that are needed to describe the mapping of $\mathcal{H}^{\underline{p}_1,\dots,\underline{p}_n}$ into itself is exactly the same. We just have to count the number of independent $b(\underline{k}_1',\underline{k}_1,\dots,\underline{k}_n',\underline{k}_n)$ where the $\underline{k}_i'$ and the $\underline{k}_i$ are both permutations of the $\underline{p}_i$. Because of eq. 117b we may restrict ourselves to the case $\underline{k}_i'=\underline{p}_i$. The permutations $\underline{k}_1,\dots,\underline{k}_n$ of $\underline{p}_1,\dots,\underline{p}_n$ and the $b(\underline{p}_1,\underline{k}_1,\dots,\underline{p}_n,\underline{k}_n)$ are in one to one correspondence.

This completes the proof that the invariant subspaces which we have found are in fact irreducible. It also shows that further dependencies among these b( )'s are not possible. Two irreducible representations in different spaces $\mathcal{H}^{\underline{p}_1,\dots,\underline{p}_n}$ and $\mathcal{H}^{\underline{q}_1,\dots,\underline{q}_n}$ are equivalent if and only if the corresponding Young graphs are equal. From this one can conclude that there cannot be further dependencies even when we take all the b( )'s into account.

For $\mathcal{B}_\varepsilon\{U[\mathcal{T}]\}'$ the b( )'s satisfy a further restriction since

$$(\Phi[T(-\underline{m})f;T(-\underline{m})g'],\mathcal{B}_\Phi[f,g]) = (\Phi[f;g'],\mathcal{B}_\Phi[T(\underline{m})f,T(\underline{m})g]) \qquad .118$$

To find this restriction we determine how both sides of this eq. differ from the r.h.s. of eq. 117a.

$(T(\underline{m})f)(\underline{x}) = f(\underline{x}-\underline{m})$ implies $(T(\underline{m})f)_{\underline{l}}(\underline{x}) = f_{\underline{l}-\underline{m}}(\underline{x}-\underline{m})$ and therefore

$$(T(\underline{m})f)(\underline{x},\underline{k}) = \sum_{\underline{l}} f_{\underline{l}-\underline{m}}(\underline{x}-\underline{m}+\underline{l})e^{-i\underline{k}\underline{l}} = e^{-i\underline{k}\underline{m}}f(\underline{x},\underline{k}) \quad .$$

Therefore we have to replace on the r.h.s. of eq. 117a $\tilde{h}(\underline{k}_i)$ by $e^{-i\underline{k}\underline{m}}\,\tilde{h}(\underline{k}_i)$ in one case and $\tilde{h}'(\underline{k}_i')$ by $e^{-i\underline{k}'\underline{m}}\,\tilde{h}'(\underline{k}_i')$ in the other. Notice that nothing goes wrong if $\tilde{h}(\underline{x},\underline{k})=h(\underline{x},\underline{k}-(2\pi,o,o))$ since $e^{+i(2\pi,o,o)\underline{m}} = 1$. Thus we find that, in order to ensure the property 118, the $b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n)$ must vanish unless

$$\sum_{j=1}^{n}(\underline{k}_j'-\underline{k}_j) = 2\pi\underline{i} \qquad \text{for some } \underline{i}\in\mathbb{Z}_3 \quad . \qquad 119$$

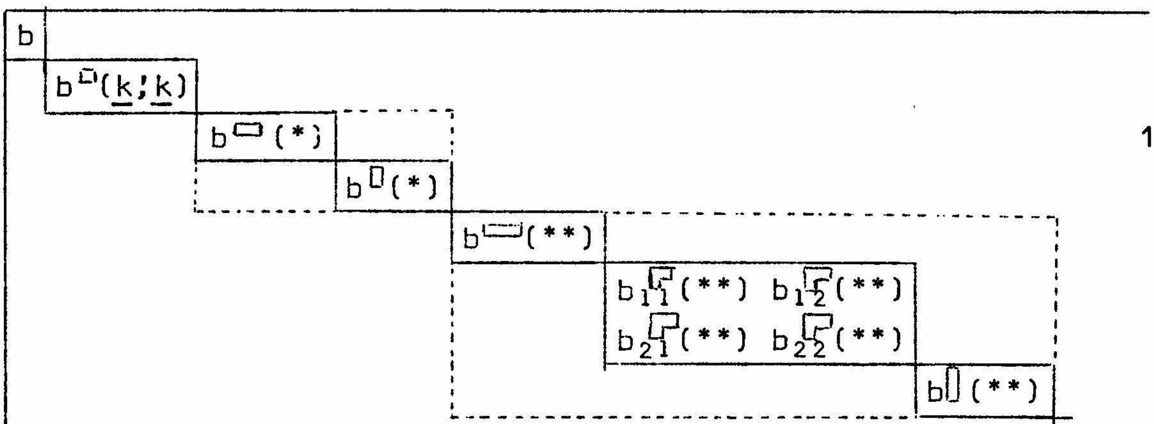
10 A Matrix Notation for the Operators $\mathcal{B}$

In this section we will introduce a notation for the operators $\mathcal{B}$ which is similar to that of eq. 41 for the RS-case. We let now each entry in the vectors describing elements of $\mathcal{H}$ correspond not just to an irreducible subspace but to the collection of all those equivalent ones that are gained by application of the same Young symmetrizer. We can refer to a Young symmetrizer by a standard tableau containing the lowest positive integers.

For the entries in the vector we will employ the notation introduced on p. 32. We give, as an example, the vector corresponding to $\Phi[f,g]$ and list besides each row the corresponding standard tableau.

$$\begin{array}{cc}
\left(\begin{array}{l}
N \\
N \quad \tilde{h}(\underline{x},\underline{k}) \\
\frac{N}{2!}\{\tilde{h}(\underline{x}_1,\underline{k}_1)\tilde{h}(\underline{x}_2,\underline{k}_2)+\tilde{h}(\underline{x}_1,\underline{k}_2)\tilde{h}(\underline{x}_2,\underline{k}_1)\}=\frac{N}{2!}\{12+21\} \\
\frac{N}{2!}\{\tilde{h}(\underline{x}_1,\underline{k}_1)\tilde{h}(\underline{x}_2,\underline{k}_2)-\tilde{h}(\underline{x}_1,\underline{k}_2)\tilde{h}(\underline{x}_2,\underline{k}_1)\}=\frac{N}{2!}\{12-21\} \\
\frac{N}{3!}\{123+231+312+132+213+321\} \\
\frac{2N}{3!}\{123-231 \qquad\qquad +213-321\} \\
\frac{2N}{3!}\{\qquad 231-312+132-213 \qquad\} \\
\frac{N}{3!}\{123+231+312-132-213-321\} \\
\qquad\qquad\vdots
\end{array}\right)
&
\begin{array}{l}
\\
\boxed{1} \\
\boxed{1\,2} \\
\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} \\
\boxed{1\,2\,3} \\
\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \\
\\
\end{array}
\end{array} \tag{120}$$

Every operator $\mathscr{B}$ can then be written as a matrix of the form



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where $(*) = (\underline{k}_1',\underline{k}_1,\underline{k}_2',\underline{k}_2)$ and $(**) = (\underline{k}_1',\underline{k}_1,\underline{k}_2',\underline{k}_2,\underline{k}_3',\underline{k}_3)$.

$(\Phi[f';g'],\mathcal{B}\Phi[f,g])$ becomes in this notation a row vector for $\Phi[f';g']$ followed by a matrix for $\mathcal{B}$ and by the column vector for $\Phi[f,g]$ if we agree that this implies matrix multiplication and integration over the $\underline{x}_i$, the $\underline{k}_i$, and over the $\underline{k}_i'$ for $\underline{k}_1' \leq \underline{k}_2' \leq \ldots \leq \underline{k}_n'$. Alternatively, we can integrate over all the values of the $\underline{k}_i'$ and divide by $\frac{1}{n!}$ if we impose a restriction similar to eq. 117b on the $b^{\boxed{graph}}(\ldots)$. If $\mathcal{B}\epsilon\{U[\mathfrak{F}]\}'$ these quantities have to satisfy also a relation similar to eq. 119. The requirement that $\mathcal{B}$ is a bounded operator places restrictions on the magnitude of the $b^{\boxed{graph}}(\ldots)$.

11 The Form of the Matrix Elements of $\mathcal{H}$

Since $e^{-it\mathcal{H}}\epsilon\ \{U[\mathfrak{F}]\}'$ it follows that

$$(\Phi[f';g'],e^{-it\mathcal{H}}\Phi[f,g]) = N'NA\{(\tilde{h}'(\underline{k}'),\tilde{h}(\underline{k}))\}$$
$$= (\Phi[f';g'],\Phi[f,g])C\{(f_i',f_j),(f_i',g_j),(g_i',f_j),(g_i',g_j)\} \quad . \qquad 122$$

The possibility of writing the matrix elements in the last form follows from eq. 114 and from the linear connection between the $\tilde{h}(\underline{x},\underline{k})$ and the $f_i(\underline{x}),g_j(\underline{x})$. If $\Phi[f,g]$ belongs to the domain where $\mathcal{H}$ is defined, we will thus be able to write

$$(\Phi[f';g'],\mathcal{H}\Phi[f,g])$$
$$= (\Phi[f';g'],\Phi[f,g])G\{(f_i',f_j),(f_i',g_j),(g_i',f_j),(g_i',g_j)\} \quad . \qquad 123$$

For real τ and $e\epsilon L_r^2(\mathbb{R}_3)$ we have

$$(\Phi[f';g'],e^{-i\tau\phi(e)}\mathcal{H}e^{i\tau\phi(e)}\Phi[f,g])$$
$$=(\Phi[f';g'],\Phi[f,g])G\{(f_i'+\tau e_i,f_j+\tau e_j),(f_i'+\tau e_i,g_j),(g_i',f_j+\tau e_j),(g_i',g_j)\} \ .$$

Under further domain restrictions we take a partial derivative with respect to τ and set $\tau=0$.

$$(\Phi[f';g'],\pi(e)\Phi[f,g]) = (\Phi[f';g'],\Phi[f,g])$$

$$\cdot\sum_{i,j}\left[\{(e_i,f_j)+(f_i',e_j)\}\frac{\partial}{\partial(f_i',f_j)}+(e_i,g_j)\frac{\partial}{\partial(f_i',g_j)}+(g_i',e_j)\frac{\partial}{\partial(g_i',f_j)}\right]$$

$$\cdot\ G\{(f_i',f_j),(f_i',g_j),(g_i',f_j),(g_i',g_j)\} \qquad 124a$$

$E=\frac{\partial}{\partial\tau}(\Phi[f';g'],\Phi[f,g+\tau e])|_{\tau=0}$ can be computed in two ways

$$E=\frac{\partial}{\partial\tau}\Bigg[e^{-\frac{i}{2}\{(f';g+\tau e)-(g';f)\}}$$

$$\cdot e^{-\sum_i\{A_i\sum_j(f_j-f_j',f_{i+j}-f_{i+j}')+C_i\sum_j(g_j+\tau e_j-g_j',g_{i+j}+\tau e_{i+j}-g_{i+j}')\}}\Bigg]\Bigg|_{\tau=0}$$

$$=\{-\tfrac{i}{2}(f';e)-2\sum_i C_i\sum_j(e_j,g_{i+j}-g_{i+j}')\}(\Phi[f';g'],\Phi[f,g])$$

and

$$E=\frac{\partial}{\partial\tau}\left[e^{\frac{i}{2}\tau(e,f)}(\Phi[f';g'],e^{-i\tau\pi(e)}\Phi[f,g])\right]\Big|_{\tau=0}$$

$$=\tfrac{i}{2}(e,f)(\Phi[f';g'],\Phi[f,g])-i(\Phi[f';g'],\pi(e)\Phi[f,g]) \quad .$$

It follows that

$$(\Phi[f';g'],\pi(e)\Phi[f,g])$$

$$=\left[(e,f+f')-4i\sum_i C_i\sum_j(e_j,g_{i+j}-g_{i+j}')\right](\Phi[f';g'],\Phi[f,g]) \quad , \qquad 124b$$

Combining this with eq. 124a we find

$$\sum_{i,j}\left[\{(e_i,f_j)+(f_i',e_j)\}\frac{\partial}{\partial(f_i',f_j)}+(e_i,g_j)\frac{\partial}{\partial(f_i',g_j)}+(g_i',e_j)\frac{\partial}{\partial(g_i',f_j)}\right]$$

$$G\{(f_i',f_j),(f_i',g_j),(g_i',f_j),(g_i',g_j)\}$$

$$=\tfrac{1}{2}(e,f+f')-2i\sum_i C_i\sum_j(e_j,g_{i+j}-g_{i+j}') \quad . \qquad 125$$

Choose $\Phi\epsilon L_r^2(\underline{o})$ with $\|\Phi\|=1$ and define $\phi_1=T(\underline{1})\Phi$. Any possible set of arguments in G{ } can be reached with $\tilde{f},\tilde{g},\tilde{f}',\tilde{g}'$ such that

$$(\phi_l,\tilde{f}_l) = (\phi_l,\tilde{g}_l) = (\phi_l,\tilde{f}'_l) = (\phi_l,\tilde{g}'_l) = 0 \qquad \forall\, l \quad .$$

Take in eq. 125 in turn

$$\begin{array}{llllll} f=\tilde{f}+\phi_k\ , & g=\tilde{g} & , f'=\tilde{f}'\ , & g'=\tilde{g}' & , e=\phi_l \\ f=\tilde{f} & , g=\tilde{g}+\phi_k\ , & f'=\tilde{f}'\ , & g'=\tilde{g}' & , e=\phi_l \\ f=\tilde{f} & , g=\tilde{g} & , f'=\tilde{f}'\ , & g'=\tilde{g}'+\phi_k\ , & e=\phi_l \quad . \end{array}$$

Note that the arguments of G{...} are in all these cases the same as if we used $f=\tilde{f},g=\tilde{g},f'=\tilde{f}',g'=\tilde{g}'$. Eq. 125 reduces to

$$\frac{\partial}{\partial(f'_l,f_k)}G\{\ldots\}=\frac{1}{2}\delta_{kl} \quad , \quad \frac{\partial}{\partial(f'_l,g_k)}G\{\ldots\}=-2iC_{k-l} \quad ,$$

$$\frac{\partial}{\partial(g'_k,f_l)}G\{\ldots\}=2iC_{k-l} \quad .$$

We conclude

$$\begin{aligned} &G\{(f'_i,f_j),(f'_i,g_j),(g'_i,f_j),(g'_i,g_j)\} \\ &= \frac{1}{2}\sum_l (f'_l,f_l)-2i\sum_{k,l} C_{k-l}\{(f'_l,g_k)-(g'_k,f_l)\}+F'\{(g'_i,g_j)\} \\ &= \frac{1}{2}\int_D \Big[(f'(\underline{k}),f(\underline{k}))-im(\underline{k})\{(f'(\underline{k}),g(\underline{k}))-(g'(\underline{k}),f(\underline{k}))\}\Big]dk \\ &\qquad +F\{(g'(\underline{k}'),g(\underline{k}))\} \end{aligned}$$

and

$$\begin{aligned} (\Phi[f;g'],H\Phi[f,g])=\frac{1}{2}\Big[&\int_D (f'(\underline{k})-im(\underline{k})g'(\underline{k}),f(\underline{k})-im(\underline{k})g(\underline{k}))dk \\ &+F\{(g'(\underline{k}'),g(\underline{k}))\}\Big]\cdot(\Phi[f;g'],\Phi[f,g]) \quad . \end{aligned} \qquad 126$$

Since $H\Phi_0=0$ it follows that

$$F\{(g'(\underline{k}'),g(\underline{k}))\equiv 0\}=0 \quad . \qquad 127$$

12 Operators that are Convenient to Construct Hamiltonians

We recognized on p. 87 that $\mathscr{h}(\underline{k})$ carries a representation of $\mathcal{T}'$ which is irreducible if $\xi(\underline{k})=1$ but which can be reduced into two equivalent irreducible representations if $\xi(\underline{k})>1$. We chose the two irreducible subspaces in such a way that their contributions to $\Phi[f,g]$ were proportional to $h(\underline{x},\underline{k})$ and to $\hbar(\underline{x},\underline{k})$. This is not the only way of splitting $\mathscr{h}(\underline{k})$. Any choice with contributions proportional to

$$H(\underline{x},\underline{k}) = h(\underline{x},\underline{k}) \cos \Theta(\underline{k}) - \hbar(\underline{x},\underline{k}) \sin \Theta(\underline{k})$$

and

$$\not{H}(\underline{x},\underline{k}) = h(\underline{x},\underline{k}) \sin \Theta(\underline{k}) + \hbar(\underline{x},\underline{k}) \cos \Theta(\underline{k})$$

with either $\sin\Theta(\underline{k})=0$ or $\cos\Theta(\underline{k})=0$ if $\xi(\underline{k})=1$ could have been employed equally well since

$$(h'(\underline{k}),h(\underline{k}))+(\hbar'(\underline{k}),\hbar(\underline{k})) = (H'(\underline{k}),H(\underline{k}))+(\not{H}'(\underline{k}),\not{H}(\underline{k})) \quad .$$

The following choice of $\Theta(\underline{k})$ will prove useful for the construction of Hamiltonians

$$\sin\Theta(\underline{k}) = \sqrt{\frac{1}{\xi(\underline{k})}} \quad , \quad \cos\Theta(\underline{k}) = \sqrt{\frac{\xi(\underline{k})-1}{\xi(\underline{k})}} = \zeta(\underline{k}) \quad .$$

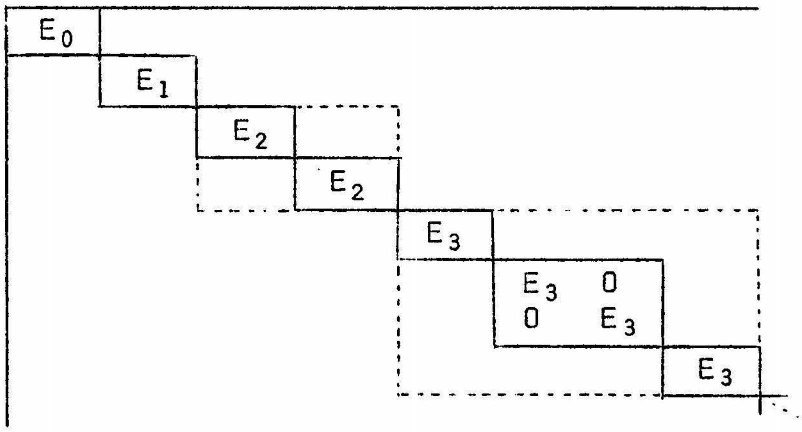
It leads to contributions proportional to

$$\tilde{d}(\underline{x},\underline{k})$$
$$= d(\underline{x},\underline{k}) = \zeta(\underline{k})h(\underline{x},\underline{k}) - \sqrt{\frac{1}{\xi(\underline{k})}}\,\hbar(\underline{x},\underline{k}) = -i\sqrt{\frac{m(\underline{k})}{2}}\,\zeta(\underline{k})g(\underline{x},\underline{k})$$

and

$$\tilde{d}(\underline{x},\underline{k}+(2\pi,0,0)) \qquad 123$$
$$= \not{d}(\underline{x},\underline{k}) = \sqrt{\frac{1}{\xi(\underline{k})}}\,h(\underline{x},\underline{k})+\zeta(\underline{k})\hbar(\underline{x},\underline{k}) = \frac{\xi(\underline{k})f(\underline{x},\underline{k})-im(\underline{k})g(\underline{x},\underline{k})}{\sqrt{2m(\underline{k})\xi(\underline{k})}} \; .$$

In the following we will frequently make use of the matrix notation 121 for operators and this often in the modified form where the h's in the vectors 120, on which the matrices act, are replaced by H's. We will consider operators that are represented by matrices of the following special form:



129

As a first example we take

$$E_n = \{\sum_{i=1}^{n} m(\underline{k}_i)\theta(\pi - k_{i_1})\}\prod_{i=1}^{n}\delta(\underline{k}'_i - \underline{k}_i) \quad , \qquad E_0 = 0 \qquad 130$$

and the vectors in their original form 120. This matrix acts in each irreducible subspace just as

$$\{\sum_{i=1}^{n} m(\underline{k}_i)\theta(\pi - k_{i_1})\}\cdot I \quad ,$$

where I is the identity operator. It presents therefore an operator, that we call $\mathcal{H}_o$, which is unbounded, selfadjoint, and nonnegative. $\mathcal{H}_o$ commutes with $U[T]$ $\forall$ $T \in \mathcal{T}$, by which we imply as usual that the domains satisfy $\mathcal{D}_{U[T]\mathcal{H}_o} = \mathcal{D}_{\mathcal{H}_o U[T]}$. $\mathcal{H}_o$ has the matrix elements

$$\begin{aligned}(\Phi[f;g'],\mathcal{H}_o\Phi[f,g]) &= N'N\Big[0 + \int_{D'} dk\ m(\underline{k})\theta(\pi - k_1)(\tilde{h}'(\underline{k}),\tilde{h}(\underline{k})) \\ &\quad + \frac{1}{(2!)^3}\int\int_{D'D'} dk_1 dk_2 \{\sum_{i=1}^{2} m(\underline{k}_i)\theta(\pi - k_{i_1})\} \\ &\cdot 2\{(\tilde{h}'(\underline{k}_1),\tilde{h}(\underline{k}_1))(\tilde{h}'(\underline{k}_2),\tilde{h}(\underline{k}_2)) + (\tilde{h}'(\underline{k}_1),\tilde{h}(\underline{k}_2))(\tilde{h}'(\underline{k}_2),\tilde{h}(\underline{k}_1)) \\ &\quad + (\tilde{h}'(\underline{k}_1),\tilde{h}(\underline{k}_1))(\tilde{h}'(\underline{k}_2),\tilde{h}(\underline{k}_2)) - (\tilde{h}'(\underline{k}_1),\tilde{h}(\underline{k}_2))(\tilde{h}'(\underline{k}_2),\tilde{h}(\underline{k}_1))\} + \dots\Big] \\ &= N'N\int_{D} m(\underline{k})(h'(\underline{k}),h(\underline{k}))dk\{1 + \int_{D'} dk(\tilde{h}'(\underline{k}),\tilde{h}(\underline{k})) + \dots\} \quad .\end{aligned}$$

Hence

$$(\Phi[f;g'],H_0\Phi[f,g]) = \int_D m(\underline{k})(h'(\underline{k}),h(\underline{k}))dk(\Phi[f;g'],\Phi[f,g]) \; . \qquad 131$$

The matrix representative for the unitary operator e^{-itH_0} is also of the form 129 but with

$$E_n = e^{-it\sum_{i=1}^{n} m(\underline{k}_i)\Theta(\pi-k_{i_1})} \prod_{i=1}^{n} \delta(\underline{k}_i'-\underline{k}_i) \quad , \quad E_0 = 1 \; .$$

Similarly as before we find

$$(\Phi[f;g'],e^{-itH_0}\Phi[f,g]) = N'Ne^{\int_{D'} dk\, e^{-itm(\underline{k})\Theta(\pi-k_1)}(\tilde{h}'(\underline{k}),\tilde{h}(\underline{k}))}$$
$$= (\Phi[f;g'],\Phi[f,g])e^{\int_D dk(e^{-itm(\underline{k})}-1)(h'(\underline{k}),h(\underline{k}))} \; . \qquad 132$$

Another example of which we will make use is

$$E_n = \{\sum_{i=1}^{n} e(\underline{k}_i)\Theta(\pi-k_{i_1})\}\prod_{i=1}^{n}\delta(\underline{k}_i'-\underline{k}_i) \quad , \quad E_0 = 0$$

and the vectors 120 with the $\tilde{h}$'s replaced by $\tilde{d}$'s. Assume that $e(\underline{k})$ is real and that it satisfies

$$\sum_{\underline{n}} \left|\int_D dk\; e(\underline{k})e^{i\underline{k}\underline{n}}\right| \quad \text{finite}$$

($m(\underline{k})$ has these properties too as stated in eq. 109).
As before we conclude that these matrices define self-adjoint operators $P(e)$ that commute with $U[T]$ $\forall\, T\epsilon\mathcal{T}$.
Their matrix elements are

$$(\Phi[f;g'],P(e)\Phi[f,g]) = \int_D e(\underline{k})(d'(\underline{k}),d(\underline{k}))dk(\Phi[f;g'],\Phi[f,g]) . \qquad 133$$

13 Existence of Hamiltonians

In this section we shall first prove that for each irreducible representation of the CCR there exists one and only one Hamiltonian $\mathcal{H}$ that satisfies the assumptions (iii)'. We shall give its matrix elements and those of $e^{-it\mathcal{H}}$. In the case of reducible representations there are many Hamiltonians $\mathcal{H}$. We shall determine the matrix elements of $\mathcal{H}$ and of $e^{-it\mathcal{H}}$ for the simplest of them.

The unitary operator $e^{-it\mathcal{H}}$ is called the time evolution operator because the time evolution of any operator that, in the Schrödinger picture, would be time independent is given by

$$\mathcal{O}(t) = e^{it\mathcal{H}}\mathcal{O}(0)e^{-it\mathcal{H}} \tag{134}$$

Eq. 117a tells us in the case of irreducible representations that $(N'N)^{-1}(\Phi[f';g'],\mathcal{B}\Phi[f,g])$ depends on $f,g,f';g'$ through the combinations $f(\underline{x},\underline{k})-im(\underline{k})g(\underline{x},\underline{k})$ and $f'(\underline{x},\underline{k})-im(\underline{k})g'(\underline{x},\underline{k})$ only. This holds in particular for $(N'N)^{-1}(\Phi[f';g'],\Phi[f,g])$ and thus for $(\Phi[f';g'],\Phi[f,g])^{-1}(\Phi[f';g'],\mathcal{B}\Phi[f,g])$ and for $(\Phi[f';g'],\Phi[f,g])^{-1}(\Phi[f';g'],\mathcal{H}\Phi[f,g])$. It follows from the eqs. 126 and 127 that

$$\begin{aligned}&(\Phi[f';g'],\mathcal{H}\Phi[f,g])\\ &= \frac{1}{2}\int_{\mathcal{D}} dk\big(f'(\underline{k})-im(\underline{k})g'(\underline{k}),f(\underline{k})-im(\underline{k})g(\underline{k})\big)\big(\Phi[f';g'],\Phi[f,g]\big)\\ &= \int_{\mathcal{D}} dk\; m(\underline{k})\big(h'(\underline{k}),h(\underline{k})\big)\big(\Phi[f';g'],\Phi[f,g]\big)\quad .\end{aligned} \tag{135a}$$

It remains to show that there exists in fact an operator with these matrix elements, that it satisfies the assumptions (iii)', and that its domain is such that the calculations in section 11 are valid. This is easily done.

The operator $\mathcal{H}_o$, defined in the previous section, is the chap we are looking for. We saw already that it is nonnegative, self-adjoint, and that it commutes with $U[T]$ $\forall$ $T\in\mathcal{T}$. The assumption eq. 111 is satisfied since $(\Phi[f;g'],\mathcal{H}_o\Phi[f,g])$ has the form of the r.h.s. of eq. 126. That $\mathcal{H}\Phi=0$ has only the solution $\Phi=c\Phi_0$ follows from the eqs. 129 and 130 because we deal with an irreducible representation of the CCR. The calculations in section 11, which led to eq. 135a, are valid since the eqs. 120, 129 and 130 inform us that $\Phi[f,g]$ lies in the domain of the operator $\mathcal{H}$ $\forall$ $f,g\in L^2_r(\mathbb{R}_3)$.

Notice that the spectrum of $\mathcal{H}$ contains a continuous part unless $m(\underline{k})$ takes on discrete values only. The matrix elements of the time evolution operator $e^{-it\mathcal{H}}$ have been determined in eq. 132 already,

$$(\Phi[f;g']e^{-it\mathcal{H}}\Phi[f,g]) = N'Ne^{\mathfrak{D}\int dk\, e^{-itm(\underline{k})}(h'(\underline{k}),h(\underline{k}))} \quad . \quad 135b$$

Guided by our experience with RS- and C_1-models, we expect that to each reducible representation of the CCR there corresponds a big variety of Hamiltonians that satisfy the assumptions (iii)'. We will not exhibit a large class of them but only derive restrictions on the functional $F\{(g'(\underline{k}'),g(\underline{k}))\}$ that appears in eq. 126 and prove the existence of Hamiltonians with $F\{...\}$ linear in $(g'(\underline{k}'),g(\underline{k}))$.

Making use of the operator $\mathcal{H}_0$ introduced in the previous section we can write $\mathcal{H}=\mathcal{H}_0+R$ with

$$(\Phi[f';g'],R\Phi[f,g]) = \frac{1}{2}F\{(g'(\underline{k}'),g(\underline{k}))\}(\Phi[f';g'],\Phi[f,g]) \qquad 136$$

Since $\mathcal{H}$ and $\mathcal{H}_0$ are hermitian and commute with $U[T]$ $\forall\, T\epsilon\mathcal{T}$, R must have these properties too. We may replace the $\tilde{h}$'s in the eqs. 117a and 114 by the $\tilde{d}$'s that we introduced in the last section. These eqs. tell us that

$$(\Phi[f';g'],R\Phi[f,g]) = (\Phi[f';g'],\Phi[f,g])$$

$$\cdot\sum_n\frac{1}{n!}\{\prod_{i=1}^{n}\iint_{\mathcal{D}'\mathcal{D}'}dk_i'dk_i(\tilde{d}'(\underline{k}_i'),\tilde{d}(\underline{k}_i))\}b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) \quad . \qquad 137$$

Comparing with 136 and keeping the definition 128 of $\tilde{d}(\underline{x},\underline{k})$ in mind we get

$$F\{(g'(\underline{k}'),g(\underline{k}))\}$$

$$=\sum_{n=1}^{\infty}\frac{2}{n!}\{\prod_{i=1}^{n}\frac{1}{2}\iint_{\mathcal{D}\mathcal{D}}dk_i'dk_i\sqrt{m(\underline{k}_i')m(\underline{k}_i)}\,\zeta(\underline{k}_i')\zeta(\underline{k}_i)(g'(\underline{k}_i'),g(\underline{k}_i))\}$$

$$\cdot\; b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) \quad .$$

Eq. 127 excludes the term $n=0$; the b's obey the relations 117b and 119. R hermitian requires the further restriction

$$b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) = b^*(\underline{k}_1,\underline{k}_1',\ldots,\underline{k}_n,\underline{k}_n') \qquad 138$$

because $(\Phi[f';g'],R\Phi[f,g])=(\Phi[f,g],R\Phi[f';g'])^*$ implies $F\{(g'(\underline{k}'),g(\underline{k}))\}=F\{(g(\underline{k}'),g'(\underline{k}))\}^*=F\{(g'(\underline{k}),g(\underline{k}'))\}^*$.
To ensure that $\mathcal{H}\Phi=0$ has only the solution $\Phi=c\Phi_0$, $F\{\;\}$ will have to fulfil some positivity condition.

Let us now investigate the special case $R=R_0$, where $F\{\;\}$ is linear in $(g'(\underline{k}'),g(\underline{k}))$. Eq. 119 requires

$b(\underline{k}';\underline{k})=e(\underline{k})\delta(\underline{k}'-\underline{k})$ and 138 restricts us to real $e(\underline{k})$. Thus we have to look for operators R_0 with matrix elements

$$(\Phi[f';g'],R_0\Phi[f,g]) = \int_D dk\ e(\underline{k})(d'(\underline{k}),d(\underline{k}))(\Phi[f';g'],\Phi[f,g]) \quad .$$

Eq. 133 assures us that for $\sum_{\underline{n}}|\int_D dk\ e(\underline{k})e^{i\underline{k}\underline{n}}|$ finite there exists such an operator and that it is self-adjoint and commutes with $U[T]$ $\forall$ $T\epsilon\mathcal{G}$. The $\Phi[f,g]$ lie in the domain of R_0 as well as in the domain of $\mathcal{H}_0$. $\mathcal{H} = \mathcal{H}_0 + R_0$ is therefore an hermitian operator. To ensure that the eq. $\mathcal{H}\Phi=0$ has up to a factor only one solution, it is sufficient that $e(\underline{k})>0$.* Since the hermitian operator $\mathcal{H}$ is nonnegative, it can be extended by the method of Friedrichs ** to a self-adjoint operator with the same lower bound. Similar arguments as in the case of irreducible representations of the CCR yield that these $\mathcal{H}$'s satisfy all our requirements.

In order to determine the matrix elements of the time evolution operator $e^{-it\mathcal{H}}$ for $\mathcal{H} = \mathcal{H}_0 + R_0$, we consider yet another possibility of splitting $\hbar(\underline{k})$, in which $\mathcal{H}$ will turn out to be diagonal. We split $\hbar(\underline{k})$ so that the contributions to $\Phi[f,g]$ from the two subspaces are propor-

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* This can be shown most easily if we use the matrix notation for $\mathcal{H}$ that we will introduce below. It follows from the fact that
$\sum_{i=1}^{n}\{m_+(\underline{k}_i)\Theta(\pi-k_{i_1})+m_-(\underline{k}_i-(2\pi,0,0))\Theta(k_{i_1}-\pi)\}$,
which appears in 142, is >0 because $e(\underline{k})>0$ implies that $m_+(\underline{k})>0$ and $m_-(\underline{k})>0$.

** See e.g. p. 329 of reference 12.

tional to

$$\tilde{H}(\underline{x},\underline{k}) = H(\underline{x},\underline{k}) = \sqrt{\rho_+(\underline{k})}\,\frac{f(\underline{x},\underline{k})-im_+(\underline{k})g(\underline{x},\underline{k})}{\sqrt{2m_+(\underline{k})}}$$

and 139

$$\tilde{H}(\underline{x},\underline{k}+(2\pi,0,0)) = \bar{H}(\underline{x},\underline{k}) = \sqrt{\rho_-(\underline{k})}\,\frac{f(\underline{x},\underline{k})-im_-(\underline{k})g(\underline{x},\underline{k})}{\sqrt{2m_-(\underline{k})}} \quad ,$$

where

$$m_\pm(\underline{k}) = \frac{1}{2}\{m(\underline{k})+e(\underline{k})\pm\sqrt{(m(\underline{k})-e(\underline{k}))^2+4m(\underline{k})e(\underline{k})\zeta^2(\underline{k})}\} \qquad 140$$

and

$$\rho_\pm(\underline{k}) = \frac{m_\pm(\underline{k})}{m_\pm(\underline{k})-m_\mp(\underline{k})}\left[1-\xi(\underline{k})\frac{m_\mp(\underline{k})}{m(\underline{k})}\right] \quad . \qquad 141$$

Easy calculations yield the relations

$$\bar{H}(\underline{x},\underline{k}) = 0 \quad \text{if} \quad \xi(\underline{k}) = 1$$

and

$$(H'(\underline{k}),H(\underline{k}))+(\bar{H}'(\underline{k}),\bar{H}(\underline{k}))$$

$$=\frac{1}{2}\{\frac{\xi(\underline{k})}{m(\underline{k})}(f'(\underline{k}),f(\underline{k}))-i(f'(\underline{k}),g(\underline{k}))+i(g'(\underline{k}),f(\underline{k}))+m(\underline{k})(g'(\underline{k}),g(\underline{k})\}$$

$$= (h'(\underline{k}),h(\underline{k}))+(\bar{h}'(\underline{k}),\bar{h}(\underline{k})) \quad ,$$

which guarantee that H and $\bar{H}$ define in fact a way of splitting $\mathfrak{h}(\underline{k})$ into irreducible subspaces.

Consider a matrix of the form 129 with

$$E_n=\sum_{i=1}^{n}\{m_+(\underline{k}_i)\Theta(\pi-k_{i_1})+m_-(\underline{k}_i-(2\pi,0,0))\Theta(k_{i_1}-\pi)\}\cdot\prod_{i=1}^{n}\delta(\underline{k}_i'-\underline{k}_i), \qquad 142$$

$$E_0=0$$

and the vectors 120 with the $\tilde{h}$'s replaced by the $\tilde{H}$'s.

The corresponding operator W has matrix elements

$$(\Phi[f';g'],W\Phi[f,g])$$

$$=(\Phi[f';g'],\Phi[f,g])\int_D dk\{m_+(\underline{k})(H'(\underline{k}),H(\underline{k}))+m_-(\underline{k})(\bar{H}'(\underline{k}),\bar{H}(\underline{k}))\}$$

$$=(\Phi[f';g'],\Phi[f,g])\int_D dk\{m(\underline{k})(h'(\underline{k}),h(\underline{k}))+e(\underline{k})(d'(\underline{k}),d(\underline{k}))\}$$

$$= (\Phi[f';g'],\mathcal{H}\Phi[f,g]) \quad .$$

In the second step of this calculation we used

$$m_+(H;H)+m_-(\mathcal{H};\mathcal{H})=\frac{1}{2}[\rho_+(f'-im_+g;f-im_+g)+\rho_-(f'-im_-g;f-im_-g)]$$
$$=\frac{1}{2}[(\rho_++\rho_-)(f;f)-i(\rho_+m_++\rho_-m_-)\{(f;g)-(g;f)\}+(\rho_+m_+^2+\rho_-m_-^2)(g;g)]$$
$$=\frac{1}{2}[(f;f)-im\{(f;g)-(g;f)\}+\{m^2+em\zeta^2\}(g;g)]=m(h;h)+e(d;d) \quad .$$

The labels $\underline{k}$ have been dropped in the above eq.

The matrix representation for $e^{-it\mathcal{H}}$ is similar to that for $\mathcal{H}$, the only difference being that now

$$E_n=e^{-it\sum_{n=1}^{n}\delta(\underline{k}_i'-\underline{k}_i)\{m_+(\underline{k}_i)\Theta(\pi-k_{i_1})+m_-(\underline{k}_i-(2\pi,0,0))\Theta(k_{i_1}-\pi)\}} \quad ,$$

$E_0=1.$

$$(\Phi[f;g'],e^{-it\mathcal{H}}\Phi[f,g])$$
$$=N'Ne^{\mathcal{D}}\int dk\; e^{-it\{m_+(\underline{k})\Theta(\pi-k_1)+m_-(\underline{k}-(2\pi,0,0))\Theta(k_1-\pi)\}}(\tilde{H}'(\underline{k}),\tilde{H}(\underline{k}))$$
$$=N'Ne^{\mathcal{D}}\int dk\{e^{-itm_+(\underline{k})}(H'(\underline{k}),H(\underline{k}))+e^{-itm_-(\underline{k})}(\mathcal{H}'(\underline{k}),\mathcal{H}(\underline{k}))\} \quad . \qquad 143$$

We collect the main results in

Theorem 2

The matrix elements of Hamiltonians $\mathcal{H}$ belonging to a reproducing kernel 109 and satisfying (iii)' have the form

$$(\Phi[f;g'],\mathcal{H}\Phi[f,g])$$
$$=\frac{1}{2}[\int_{\mathcal{D}}(f'(\underline{k})-im(\underline{k})g'(\underline{k}),f(\underline{k})-im(\underline{k})g(\underline{k}))dk+F\{(g'(\underline{k}'),g(\underline{k}))\}]$$
$$\cdot\;(\Phi[f;g'],\Phi[f,g]) \qquad 144$$

with $\qquad F\{(g'(\underline{k}'),g(\underline{k}))\}$

$$=\sum_{n=1}^{\infty}\frac{2}{n!}\{\prod_{i=1}^{n}\frac{1}{2}\int_{\mathcal{D}}\int_{\mathcal{D}}dk_i'dk_i\sqrt{m(\underline{k}_i')m(\underline{k}_i)}\,\zeta(\underline{k}_i')\zeta(\underline{k}_i)(g'(\underline{k}_i'),g(\underline{k}_i))\}$$
$$\cdot\;b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) \quad ,$$

where

$$b(\underline{k}_1',\underline{k}_1,\ldots,\underline{k}_n',\underline{k}_n) = \begin{cases} b(\underline{k}_{i_1}',\underline{k}_{i_1},\ldots,\underline{k}_{i_n}',\underline{k}_{i_n}) & \forall \text{ permutations } i_1,\ldots,i_n \text{ of } 1,\ldots,n \\ 0 & \text{unless } \sum_{j=1}^{n}(\underline{k}_j'-\underline{k}_j) = 2\pi\underline{i} \\ b^*(\underline{k}_1,\underline{k}_1',\ldots,\underline{k}_n,\underline{k}_n') & . \end{cases}$$

We proved that such $\mathcal{H}$ exist whenever

$$(\Phi[f;g'],\mathcal{H}\Phi[f,g]) = \frac{1}{2}\int_{D}\{(f'(\underline{k})-im(\underline{k})g'(\underline{k}),f(\underline{k})-im(\underline{k})g(\underline{k})) + e(\underline{k})m(\underline{k})\zeta^2(\underline{k})(g'(\underline{k}),g(\underline{k}))\}dk(\Phi[f;g'],\Phi[f,g]) \quad , \qquad 145$$

$e(\underline{k})>0$, and $\sum_{\underline{n}}|\int_{D}dk\ e(\underline{k})e^{i\underline{k}\underline{n}}|$ finite.

Let us denote by F_o the functional F from which we have split off the linear terms in $(g'(\underline{k}'),g(\underline{k}))$. The diagonal matrix elements of $\mathcal{H}$ have the form

$$(\Phi[f,g],\mathcal{H}\Phi[f,g])$$

$$= \frac{1}{2}\left[\int_{D}\{(f(\underline{k}),f(\underline{k}))+m_o^2(\underline{k})(g(\underline{k}),g(\underline{k}))\}dk+F_o\{(g(\underline{k}'),g(\underline{k}))\}\right] \quad ,$$

where

$$m_o^2(\underline{k}) = m^2(\underline{k})+e(\underline{k})m(\underline{k})\zeta^2(\underline{k}) = \rho_+(\underline{k})m_+^2(\underline{k})+\rho_-(\underline{k})m_-^2(\underline{k}) \quad .$$

14 The Two-point Function *

First we consider irreducible representations of the CCR. We will use in the following the notation

$$\phi(f,t) = e^{it\mathcal{H}}\phi(f)e^{-it\mathcal{H}} \quad .$$

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* I would like to thank Prof. J.R. Klauder for correspondence on the questions dealt with in this section.

In order to compute

$$(\Phi_0,\phi(f,t')\phi(g,t'')\Phi_0)=(\Phi_0,\phi(f)e^{-itH}\phi(g)\Phi_0)=(\Phi_0,\phi(f,t)\phi(g)\Phi_0) \quad ,$$

where $t=t'-t''$, we consider

$$(\Phi_0, e^{-i\phi(f)}e^{-itH}e^{i\phi(g)}\Phi_0) \underset{135b}{=} N'N e^{\int_D dk\, e^{-itm(\underline{k})}\frac{(f(\underline{k}),g(\underline{k}))}{2m(\underline{k})}} \quad .$$

Equating the terms that are linear in both f and g we get

$$(\Phi_0,\phi(f)e^{-itH}\phi(g)\Phi_0) = \int_D dk\frac{e^{-itm(\underline{k})}}{2m(\underline{k})}(f(\underline{k}),g(\underline{k})) \quad . \qquad 146$$

If $f(\underline{x})$ is tame enough we can Fourier-transform it

$$f(\underline{x})=\int dk\, \tilde{f}(\underline{k})e^{i\underline{k}\underline{x}} \quad , \quad \tilde{f}(\underline{k})=\int d\underline{x} f(\underline{x})e^{-i\underline{k}\underline{x}} \quad .$$

Both integrations extend over $\mathbb{R}_3$, $dk=\frac{dk_1dk_2dk_3}{(2\pi)^3}$, and $d\underline{x}=dx_1dx_2dx_3$. We call $\underline{k}$ the momentum and introduce the reduced momentum $\underline{k}_r\varepsilon D$, $\underline{k}_r+2\pi\underline{n}=\underline{k}$ $(\underline{n}\varepsilon\mathbb{Z}_3)$.

$$\tilde{f}(\underline{k})=\int_0 d\underline{x}\sum_{\underline{l}} f(\underline{x}+\underline{l})e^{-i\underline{k}(\underline{x}+\underline{l})}=\int_0 d\underline{x}e^{-i\underline{k}\underline{x}}\sum_{\underline{l}}\underbrace{e^{-i2\pi\underline{n}\underline{l}}}_{1}e^{-i\underline{k}_r\underline{l}}f(\underline{x}+\underline{l})$$

$$= \int_0 d\underline{x}e^{-i\underline{k}\underline{x}}f(\underline{x},\underline{k}_r) \quad ,$$

i.e.

$$\tilde{f}(\underline{k})=\int_0 d\underline{x}e^{-i\underline{k}\underline{x}}f(\underline{x},\underline{k}_r) \quad .$$

The inverse transformation is

$$f(\underline{x},\underline{k}_r) = \sum_{\underline{l}}\tilde{f}(\underline{k}_r+2\pi\underline{l})e^{i(\underline{k}_r+2\pi\underline{l})\underline{x}} \quad .$$

The eq. 146 becomes

$$(\Phi_0,\phi(f)e^{-itH}\phi(g)\Phi_0)$$

$$=\int_D dk\frac{e^{-itm(\underline{k})}}{2m(\underline{k})}\int_0 d\underline{x}'\sum_{l}\tilde{f}^*(\underline{k}+2\pi\underline{l})e^{-i(\underline{k}+2\pi\underline{l})\underline{x}'}\sum_{\underline{m}}\tilde{g}(\underline{k}+2\pi\underline{m})e^{i(\underline{k}+2\pi\underline{m})\underline{x}'}$$

$$=\int dk\frac{e^{-itm(\underline{k}_r)}}{2m(\underline{k}_r)}\int_0 d\underline{x}'\tilde{f}^*(\underline{k})e^{-i\underline{k}\underline{x}'}\sum_{\underline{n}}\tilde{g}(\underline{k}+2\pi\underline{n})e^{i(\underline{k}+2\pi\underline{n})\underline{x}'} \quad . \qquad 147$$

In the last step we assumed f tame enough that the change in the order of $\underline{l}$ summation and $\underline{x}'$ integration is valid. Let us extend the domain where $m(\underline{k})$ and $\xi(\underline{k})$ are defined to all $\mathbb{R}_3$ by putting $m(\underline{k})=m(\underline{k}_r)$ and $\xi(\underline{k})=\xi(\underline{k}_r)$. We boldly replace in the expression on the r.h.s. of eq. 147 $f(\underline{x}')$ by $\delta(\underline{x}'-\underline{x})$ and $g(\underline{x}')$ by $\delta(\underline{x}'-\underline{y})$ (i.e. $\tilde{f}(\underline{k})=e^{-i\underline{k}\underline{x}}$, $\tilde{g}(\underline{k})=e^{-i\underline{k}\underline{y}}$), call the resulting expression the two-point function, and use for it the familiar notation $\langle 0|\phi(\underline{x})e^{-itH}\phi(\underline{y})|0\rangle = \langle 0|\phi(\underline{x},t)\phi(\underline{y})|0\rangle$.

$$\langle 0|\phi(\underline{x},t)\phi(\underline{y})|0\rangle = \int dk\frac{e^{-itm(\underline{k})}}{2m(\underline{k})}\int_{\circ} d\underline{x}' e^{i\underline{k}(\underline{x}-\underline{x}')}\sum_{\underline{n}} e^{-i(\underline{k}+2\pi\underline{n})(\underline{y}-\underline{x}')}$$

$$=\int dk\frac{e^{-itm(\underline{k})+i\underline{k}(\underline{x}-\underline{y})}}{2m(\underline{k})}\int_{\circ} d\underline{x}'\sum_{\underline{n}} e^{-2\pi i\underline{n}(\underline{y}-\underline{x}')} \quad .$$

$$\int_{\circ} d\underline{x}'\sum_{\underline{n}} e^{-2\pi i\underline{n}(\underline{y}-\underline{x}')} = \int_{\circ} d\underline{x}'\sum_{\underline{s}}\delta(\underline{y}-\underline{x}'+\underline{s}) = \int d\underline{x}'\,\delta(\underline{y}-\underline{x}') = 1$$

Thus we get finally

$$\langle 0|\phi(\underline{x},t)\phi(\underline{y})|0\rangle = \int dk\frac{e^{i\{\underline{k}(\underline{x}-\underline{y})-m(\underline{k})t\}}}{2m(\underline{k})} \quad . \qquad 148$$

This two-point function looks formally very similar to the one for a relativistic free field (RFF), which can be found in numerous textbooks, e.g. on p. 33 of reference 14:

$$\langle 0|\phi(\underline{x},t)\phi(\underline{y})|0\rangle_{RFF} = \int dk\frac{e^{i\{\underline{k}(\underline{x}-\underline{y})-\omega_{\underline{k}}t\}}}{2\omega_{\underline{k}}} \quad .$$

The difference between the two expressions is that $m(\underline{k})$ is periodic in k_1, k_2, and k_3 whereas $\omega_{\underline{k}}=\sqrt{m^2+\underline{k}^2}$ depends on $|\underline{k}|$ only, the values of $|\underline{k}|$ and $\omega_{\underline{k}}$ being in one to one correspondence.

In the case of reducible representations of the CCR we have

$$(\Phi_0, e^{-i\phi(f)}e^{-itH}e^{i\phi(g)}\Phi_0) \stackrel{143}{=}$$

$$= e^{\int_D dk\{e^{-itm_+(\underline{k})}\frac{\rho_+(\underline{k})}{2m_+(\underline{k})}(f(\underline{k}),g(\underline{k}))+e^{-itm_-(\underline{k})}\frac{\rho_-(\underline{k})}{2m_-(\underline{k})}(f(\underline{k}),g(\underline{k}))\}}$$

This is true if $H=H_0+R_0$. However, the relation that we get from it by equating the terms which are linear in both f and g is generally true since higher order terms in F{ } do not contribute.

$$(\Phi_0, \phi(f)e^{-itH}\phi(g)\Phi_0)$$

$$= \int_D dk\{\frac{\rho_+(\underline{k})}{2m_+(\underline{k})}e^{-itm_+(\underline{k})}+\frac{\rho_-(\underline{k})}{2m_-(\underline{k})}e^{-itm_-(\underline{k})}\}(f(\underline{k}),g(\underline{k})) \quad .$$

Proceeding as before we find

$$\langle 0|\phi(\underline{x},t)\phi(\underline{y})|0\rangle$$

$$= \int dk e^{i\underline{k}(\underline{x}-\underline{y})}\{\frac{\rho_+(\underline{k})}{2m_+(\underline{k})}e^{-im_+(\underline{k})t}+\frac{\rho_-(\underline{k})}{2m_-(\underline{k})}e^{-im_-(\underline{k})t}\} \quad . \qquad 149$$

15 Discussion and Physical Interpretation of C_2-Models

Our convention to call $\underline{k}$ the momentum is justified by the way in which it enters eq. 149.

$\rho_+(\underline{k})$ and $\rho_-(\underline{k})$ are both >0 if $\xi(\underline{k})>1$. It follows for a particle with a definite value of the energy and with a momentum $\underline{k}$ for which $\xi(\underline{k})>1$ that two different values of the energy are possible. The energies $m_\pm(\underline{k})$ are periodic functions of the momentum $\underline{k}$. The two states corresponding to a given $\underline{k}$ are created by the field with relative strength $\rho_+(\underline{k})\geq 0$ and $\rho_-(\underline{k})\geq 0$, $\rho_+(\underline{k})+\rho_-(\underline{k})=1$. If we make the transition $\xi(\underline{k})\to 1$, then $\rho_+(\underline{k})\to 0$ for $e(\underline{k})>m(\underline{k})$, $\rho_-(\underline{k})\to 0$ for $e(\underline{k})<m(\underline{k})$, and $\rho_+(\underline{k})$, $\rho_-(\underline{k})\to\frac{1}{2}$ for $e(\underline{k})=m(\underline{k})$.

States that contain both kinds of particles satisfy generalized Bose statistics because all the vectors in $\mathcal{H}$ are left invariant if we permute the sets of coordinates that refer to the various particles; i.e. the sign does not change if we permute the whole sets. This feature appeared in RS-models already.

A problem of great interest concerns the S-operator. We write $\mathsf{H}=\mathsf{H}_0+V$, where H_0 is the operator whose matrix elements were given in eq. 131. Note that H_0 and V act in the same Hilbert space as H. Therefore it makes sense to define

$$S = \lim_{\substack{t_1\to+\infty \\ t_2\to-\infty}} e^{i\mathsf{H}_0 t_1} e^{-i\mathsf{H} t_1} e^{i\mathsf{H} t_2} e^{-i\mathsf{H}_0 t_2} \qquad 150$$

We would like the limes to exist in the strong sense and S to turn out to be unitary. A sufficient condition for strong convergence is e.g.

$$\int_{-\infty}^{+\infty} \|Ve^{-i\mathsf{H}_0 t}\psi\|\,dt<\infty \qquad \forall\ \psi \text{ of a total set} \quad ,$$

and we may ask for what H it is satisfied.

16 A Special Model

At an earlier stage of my investigations, when I had not yet recognized the limitations common to all the C_2-models, my imagination was fired by the special model that I am going to describe, which looked at first sight much closer to relativistic than any RS- or C_1-model.

The classical Hamiltonian for a relativistic free field was given in eq. 2 already,

$$H(f,g) = \frac{1}{2}\int\{f^2(\underline{x})+(\nabla g)^2(\underline{x})+m^2g^2(\underline{x})\}d\underline{x} \qquad 151$$

We approximate ∇g as follows. Consider a division of space into unit cubes, which is a special example for the kind of divisions proposed in the introduction to chapter VI. We replace $\nabla g(\underline{x})$ by $\left(g(\underline{x}+\underline{e}_1)-g(\underline{x}),g(\underline{x}+\underline{e}_2)-g(\underline{x}),g(\underline{x}+\underline{e}_3)-g(\underline{x})\right)$. where $\underline{e}_i$ is a unit vector along the i'th coordinate axis. This should be a good approximation if we choose the length unit of the order of 10^{-16}cm. It leads to the following approximation for H(f,g).

$$H_a(f,g)=\frac{1}{2}\Big[(f,f)+2\sum_{\underline{i}\in\mathbb{Z}_3}(g_{\underline{i}},3g_{\underline{i}}-g_{\underline{i}+\underline{e}_1}-g_{\underline{i}+\underline{e}_2}-g_{\underline{i}+\underline{e}_3})+m^2(g,g)\Big] \qquad 152$$

We want to show that there exists a C_2-model corresponding to an irreducible representation of the CCR whose Hamiltonian has diagonal matrix elements that coincide with 152. That there exists an irreducible representation with this property, is a very desirable feature since we want to mimic a free field. 135a, which holds in the irreducible case, tells us that

$$(\Phi[f,g],H\Phi[f,g])=\frac{1}{2}[(f,f)+\int_{D}dk\ m^2(\underline{k})(g(\underline{k}),g(\underline{k}))]$$

The right hand sides of this and of 152 coincide if we choose

$$m(\underline{k}) = \sqrt{6-e^{ik_1}-e^{-ik_1}-e^{ik_2}-e^{-ik_2}-e^{ik_3}-e^{-ik_3}+m^2} .$$

We have to show that $m(\underline{k})$ satisfies the assumptions that went into the derivation of theorem 1b on p. 98. $m(\underline{k})>0$ is obvious, but are the sums Σ_1 and Σ_2 of the absolute values of the Fouriercoefficients of $m(\underline{k})$ and of $\frac{1}{m(\underline{k})}$ finite? We expand

$$m(\underline{k}) = \sqrt{6+m^2}\sqrt{1-\frac{\cos k_1+\cos k_2+\cos k_3}{3+\frac{m^2}{2}}}$$

into a power series using that

$$\sqrt{1-x} = 1-\frac{1}{2}x-\frac{1!!}{2^2}\frac{x^2}{2!}-\frac{3!!}{2^3}\frac{x^3}{3!}-\frac{5!!}{2^4}\frac{x^4}{4!}-\dots \qquad \text{for } |x|\leqslant 1$$

With the help of relations like

$$\cos k_1 \cos k_2 = \frac{1}{2}[\cos(k_1-k_2)+\cos(k_1+k_2)] \quad ,$$

$$\cos k_1 \cos k_2 \cos k_3$$

$$=\frac{1}{4}[\cos(k_1+k_2-k_3)+\cos(k_2+k_3-k_1)+\cos(k_3+k_1-k_2)+\cos(k_1+k_2+k_3)],\dots$$

we can conclude that

$$\Sigma_1 = \sqrt{6+m^2}\left[1+\frac{1}{2}\left[\frac{3}{3+\frac{m^2}{2}}\right]+\frac{1!!}{2^2\cdot 2!}\left[\frac{3}{3+\frac{m^2}{2}}\right]^2+\frac{3!!}{2^3\cdot 3!}\left[\frac{3}{3+\frac{m^2}{2}}\right]^3+\dots\right]$$

$$= \sqrt{6+m^2}\left[2-\sqrt{1-\frac{3}{3+\frac{m^2}{2}}}\right]$$

Similarly we find using

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3!!}{2^2}\frac{x^2}{2!} + \frac{5!!}{2^3}\frac{x^3}{3!} + \dots \quad \text{for } |x|<1$$

that

$$\Sigma_2 = \frac{1}{\sqrt{6+m^2}} \frac{1}{\sqrt{1-\frac{3}{3+\frac{m^2}{2}}}} .$$

For small values of $|\underline{k}|$ we obtain

$$m(\underline{k}) = \sqrt{m^2+k_1^2+k_2^2+k_3^2} \approx m+\frac{|\underline{k}|^2}{2m} ,$$

which is the nonrelativistic energy-momentum relationship. We may choose our length unit so small and, consequently, our momentum unit so big that the above approximation is good as long as the nonrelativistic approximation to the relativistic energy-momentum relationship is.

As far as this special model gives results that bear analogy to properties of the relativistic free field, it will be interesting to study the corresponding results for models with the same $m(\underline{k})$ but $\xi(\underline{k})>1$ and with a suitable Hamiltonian. These results will give us some indication as to what properties we will have to expect of self-interacting relativistic field theories.

APPENDIX A

The Silver Rule of Weak Convergence

Let B_n be a sequence of uniformly bounded operators, $\|B_n\|\leq k$, and let $C_n\{\lambda_1,\lambda_2\}$ with $\lambda_1,\lambda_2\varepsilon\mathcal{H}$ be the linear functional defined by $C_n\{\lambda_1,\lambda_2\}=(\lambda_1,B_n\lambda_2)$. The theorem, which we call "the silver rule of weak convergence", states:

$$\boxed{\begin{array}{l}\text{If } C\{\phi,\psi\}=\lim\limits_{n\to\infty} C_n\{\phi,\psi\} \text{ exists } \forall\ \phi,\psi \text{ in a total set } \mathcal{T}, \\ \text{then } B_n \rightharpoonup B \text{ and } (\phi,B\psi)=C\{\phi,\psi\}.\end{array}} \qquad \text{A1}$$

A set $\mathcal{T}$ of elements of a space is called total if the elements of $\mathcal{T}$ span that space. In all our models we assumed that the $\Phi[f,g]$ form a total set with respect to $\mathcal{H}$.

It is well known that A1 is correct if we replace in it $\phi,\psi\varepsilon\mathcal{T}$ by $\lambda_1,\lambda_2\varepsilon\mathcal{H}$. A proof is given on p. 61 f. of reference 15 e.g. Thus it remains to show that the existence of $\lim\limits_{n\to\infty} C_n\{\lambda_1,\lambda_2\}$ for $\lambda_1,\lambda_2\varepsilon\mathcal{T}$ implies its existence $\forall\ \lambda_1,\lambda_2\varepsilon\mathcal{H}$. Let $\lambda_1\varepsilon\mathcal{H},\lambda_2\varepsilon\mathcal{H},\delta_1>0,\delta_2>0$ be given. There exist

$$\Phi=\sum_{n=1}^{N_{\lambda_1}(\delta_1)} c_n\phi_n \quad \text{and} \quad \Psi=\sum_{n=1}^{N_{\lambda_2}(\delta_2)} d_n\psi_n \quad , \quad \phi_i,\psi_i\varepsilon\mathcal{T} ,$$

such that $\|\lambda_1-\Phi\|<\delta_1$ and $\|\lambda_2-\Psi\|<\delta_2$ because the finite linear combinations of a total set form a dense set. It follows from

$$(\lambda_1,B_n\lambda_2)=\big(\lambda_1-\Phi,B_n(\lambda_2-\Psi)\big)+(\lambda_1-\Phi,B_n\Psi)+\big(\Phi,B_n(\lambda_2-\Psi)\big)+(\Phi,B_n\Psi)$$

that

$$|(\lambda_1,B_n\lambda_2)-(\Phi,B_n\Psi)|\leqslant k\delta_1\delta_2+k\delta_1\|\Psi\|+k\|\Phi\|\delta_2 \quad . \qquad \text{A2}$$

The r.h.s. does not depend on n and can be made as small as we like by appropriate choices of δ_1 and δ_2. The convergence of the $C_n\{\phi,\psi\}$ $\forall\ \phi,\psi\varepsilon\mathfrak{T}$ implies the convergence of $(\Phi,B_n\Psi)$ and,thanks to A2, also of $(\lambda_1,B_n\lambda_2)$, where λ_1 and λ_2 are arbitrary elements of $\mathcal{H}$.

APPENDIX B

Remarks on a Conjecture

On p. 87 we made the conjecture that the statement "necessary and sufficient for

$$(\Phi_0, \Phi[f,g]) = e^{-\int_D dk \int_0 d\underline{x} \{A(\underline{k})|f(\underline{x},\underline{k})|^2 + C(\underline{k})|g(\underline{x},\underline{k})|^2\}} \qquad \text{B1}$$

with piecewise constant $A(\underline{k})$ and $C(\underline{k})$ to correspond to $U[f,g]$ satisfying (i) is that $A(\underline{k})C(\underline{k}) \geq \frac{1}{16}$ $\forall$ $\underline{k} \in D$; the representation is irreducible if $A(\underline{k})C(\underline{k}) = \frac{1}{16}$ $\forall$ $\underline{k} \in D$ and reducible otherwise" holds for all $A(\underline{k}), C(\underline{k}) \in L^2(D)$. In a letter, Prof. J.R. Klauder outlined the following simple and elegant proof of our conjecture for the case $A(\underline{k})C(\underline{k}) = \frac{1}{16}$.

Imagine the assumptions (i) and (ii) written for the $f(\underline{x},\underline{k})$ that correspond to $f(\underline{x}) \in L$ instead of writing them, as we did, for the $f \in L$. Only eq. 73 will look different:

$$U[f',g']U[f,g]$$

$$= e^{\frac{i}{2}\int_D dk \int_0 d\underline{x} \{f'^*(\underline{x},\underline{k})g(\underline{x},\underline{k}) - g'^*(\underline{x},\underline{k})f(\underline{x},\underline{k})\}} U[f'+f, g'+g]$$

We define

$$F(\underline{x},\underline{k}) = \frac{1}{2\sqrt{C(\underline{k})}} f(\underline{x},\underline{k}) \quad ,$$

$$G(\underline{x},\underline{k}) = 2\sqrt{C(\underline{k})}\; g(\underline{x},\underline{k}) \quad ,$$

and

$$U[f,g] = U_0[F,G] = U_0\left[\frac{1}{2\sqrt{C}} f, 2\sqrt{C} g\right]$$

One easily convinces oneself that the following two sets of assumptions are equivalent: 1) the $U[f,g]$ satisfy (i) $\forall$ $f(\underline{x},\underline{k})$ and $g(\underline{x},\underline{k})$ that correspond to $f,g\epsilon L$, and 2) the $U_0[F,G]$ satisfy the analogous requirements (I) for all F and G that correspond to $f,g\epsilon L$.

B1 can be written as

$$(\Phi_0, U_0[F,G]\Phi_0) = e^{-\frac{1}{4}\int dk \int_0 d\underline{x} \{\xi(\underline{k})|F(\underline{x},\underline{k})|^2 + |G(\underline{x},\underline{k})|^2\}}$$

because $\xi(\underline{k})=16A(\underline{k})C(\underline{k})$. If $\xi(\underline{k})\equiv 1$, then one can prove, using the same methods as Klauder in reference 1, that there exist $U_0[F,G]$'s that satisfy (I) and that the $U_0[F,G]\Phi_0$ span a space containing an irreducible representation of the CCR. Therefore the $U[f,g]$ fulfil (i) and define an irreducible representation of the CCR.

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